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ORDINARY DIFFERENTIAL EQUATIONS

Representation of Solutions of Hybrid Difference-Differential Systems

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INTRODUCTION

In the investigation of real physical processes, one deals with both dynamic (differential) and algebraic (functional) dependencies. Such processes are described by differential-algebraic systems [1] (some equations in them are differential, and the other are algebraic) or mixed difference-differential systems. One classifies them as hybrid systems [2–5]. However, note that the term "hybrid systems" is overloaded. Nowadays, especially in publications in English, this term is used mainly in connection with discrete-continuous systems or systems containing logical variables [6–8]. In general, hybridity means inhomogeneity in the nature of the process to be considered or in the methods to be used in the analysis. The term "hybrid systems" pertains to systems describing processes or objects with substantially different characteristics, for examples, containing continuous and discrete variables (signals) in the main dynamics, deterministic and random variables or actions, and so on, which eventually specifies the character (nature) of hybrid systems. There are numerous examples of hybrid systems. In control theory, there is a well known representative of hybrid systems, namely, a linear continuous autonomous object described by linear differential equations (the mathematical model is based on a recording device that operates continuously) and controlled by a discrete linear autonomous controller described by finite-difference equations (a recording device that operates discretely is used). These types of systems are usually analyzed on layers and are known as discrete data systems or digital control systems. Another standard example of a hybrid control system is a commutation system, where the behavior can be described by finitely many dynamical models (systems of differential or finite-difference equations) together with a list of rules for switching between these models. One more direction in the theory of hybrid systems is related to the analysis of qualitative properties (like stability) of dynamical systems described by difference-differential equations with discontinuous coefficients, i.e., systems with variable dynamic structure. A classical practical example of a hybrid system is provided by the heating–cooling system of a dwelling house. A heater and a conditioner, together with the characteristics of the heat flow, form a system to be controlled. A thermostat is a discretely controlled system, which mainly processes symbols "too hot," "too cool," and "normal." There are numerous reasons to use hybrid models: the adequacy of these models, their justified simplification, the use of digital devices (a control with the use of computer software); hybrid systems arise in the simulation of the hierarchical structure of real control systems, in particular, in the description of dynamical, discrete, stochastic subsystems, complex systems, and so on. For more information on hybrid systems, see [2–10].

In the present paper, we consider algebraic-differential delay systems to which, in particular, some standard types of linear discrete-continuous systems and systems with retarded argument of neutral type can be reduced. Such systems can be qualified as hybrid difference-differential systems or completely regular algebraic-differential systems with delay [11], which are, in turn, a special case of descriptor (singular) systems with aftereffect

$$\frac{d}{dt}\left(\int_{-h}^{0} d_s G(t,s)x(t+s) + \int_{-h}^{0} d_s Q(t,s)u(t+s) + F_1(t)\right)$$
$$= \int_{-h}^{0} d_s A(t,s)x(t+s) + \int_{-h}^{0} d_s B(t,s)u(t+s) + F_2(t),$$

where an n-vector function $x(\cdot)$ describes the behavior in time of the object (process) to be modeled, $u(\cdot)$ is an r-vector function specifying the input influence (control), the n-vector functions $F_1(\cdot)$ and $F_2(\cdot)$ specify perturbations, the entries of the matrix functions $G(t,s)$, $Q(t,s)$, $A(t,s)$, and $B(t,s)$ of the corresponding size have a bounded variation with respect to the second argument on $[-h,0]$ and $h > 0$ is the value of the aftereffect.

In the stationary case of this equation with an operator $G : C([-h,0],\mathbb{R}^n) \to \mathbb{R}^n$ atomic at zero, the study of the existence, uniqueness, exponential estimate, and stability of solutions as well as their representation by the variation-of-constants formula can be found in [12, 13]. All these problems remain open in the general case for nonatomic operators.

In what follows, we consider properties of solutions of a special case of the above-represented schemes, namely, linear hybrid difference-differential systems with numerous delays in the state and the control. The results of the paper were earlier announced in [14].

1. INITIAL VALUE PROBLEM

Consider an object whose mathematical model of the motion is governed by the hybrid difference-differential system

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=0}^{l} A_{11i}(t)x(t-ih) + \sum_{i=0}^{l} A_{12i}(t)y(t-ih) + \sum_{i=0}^{l} B_{1i}(t)u(t-ih),\\
y(t) &= \sum_{i=0}^{l} A_{21i}(t)x(t-ih) + \sum_{i=0}^{l} A_{22i}(t)y(t-ih) + \sum_{i=0}^{l} B_{2i}(t)u(t-ih), \qquad t \ge t_0,
\end{aligned} \tag{1.1}$$

with the initial conditions

$$\begin{gathered}
x(t_0+0) = x(t_0) = x_0 \in \mathbb{R}^n,\\
x(\tau) = \varphi(\tau), \qquad y(\tau) = \psi(\tau), \qquad u(\tau) = \xi(\tau), \qquad \tau \in (-\infty, t_0);\\
\varphi(\tau) = 0, \qquad \psi(\tau) = 0, \qquad \xi(\tau) = 0, \qquad \tau \in (-\infty, t_0 - lh).
\end{gathered} \tag{1.2}$$

Here $l \in \mathbb{N}$, $A_{220}(t) \equiv 0$, and the entries of the matrix functions $A_{11i}(t) \in \mathbb{R}^{n\times n}$, $A_{12i}(t) \in \mathbb{R}^{n\times m}$, $A_{21i}(t) \in \mathbb{R}^{m\times n}$, $A_{22i}(t) \in \mathbb{R}^{m\times m}$, $B_{1i}(t) \in \mathbb{R}^{n\times r}$, and $B_{2i}(t) \in \mathbb{R}^{m\times r}$ $(i = 0,1,\dots,l)$ are piecewise continuous functions for $t \in [t_0 - lh, +\infty)$.

The functions

$$\psi(\cdot) \in PC((-\infty,t_0],\mathbb{R}^m), \qquad \varphi(\cdot) \in PC((-\infty,t_0],\mathbb{R}^n), \qquad \xi(\cdot) \in PC((-\infty,t_0],\mathbb{R}^r)$$

are admissible initial data. An admissible control is a vector function $u(\cdot) \in PC([t_0,+\infty),\mathbb{R}^r)$. The symbol $PC(\Delta,\mathbb{R}^n)$ stands for the set of piecewise continuous n-vector functions on the interval Δ.

In the stationary case, the matrices occurring in (1.1) and (1.2) are constant, and the symbol t is omitted:

$$\begin{gathered}
A_{mki}(t) = A_{mki}, \qquad B_{mi}(t) = B_{mi},\\
m,k = 1,2, \qquad i = 0,1,\dots,l, \qquad t \ge -lh, \qquad t_0 = 0.
\end{gathered} \tag{1.3}$$

Definition 1.1. A *solution* $x(t) = x(t;t_0,x_0,\varphi,\psi,\xi,u)$, $y(t) = y(t;t_0,x_0,\varphi,\psi,\xi,u)$, $t \ge t_0$, of system (1.1) with the initial conditions (1.2) and an admissible control $u = u(t)$, $t \ge t_0$, is defined as arbitrary vector functions $x(t)$ and $y(t)$, $t \geq t_0$, satisfying the first equation of the system for $t \geq t_0$, $t - t_0 \neq kh$, $k = 0, 1, \dots$, and the second equation of the system for $t \geq t_0$; moreover, the vector function $x(\cdot)$ is assumed to be piecewise smooth and continuous, and $y(\cdot)$ is assumed to be piecewise continuous on the interval $[t_0, +\infty)$.

2. INTEGRAL REPRESENTATION OF SOLUTIONS ON THE BASIS OF ADJOINT SYSTEMS

The role of the Cauchy problem in the theory of dynamical systems was noted in [15, p. 227]: "As a rule, the Cauchy problem, that is, determination of the phase trajectory for given controls, given perturbations, and given initial conditions is a key point in practically all simulation procedures." In what follows, for nonstationary systems, we derive formulas representing solutions via the solutions of the corresponding adjoint systems, which generalizes the variation-of-constants formula, well-known for ordinary systems, to the case of such systems. To derive such representations, we use the classical ideas due Bellman and Cooke [12], but the adjoint systems and hence their solutions require a substantial modification for our class of problems. The presence of a jump equation in the adjoint system is essential in this connection. The results are refined in the stationary case, which, in particular, permits one to consider the natural direction of time in the adjoint system. The special case of hybrid difference-differential systems given by simplest systems in the normal form with a single delay was considered in [16].

Let matrix functions $X(t,\tau)$, $Z(t,\tau)$, and $Y(t,\tau)$ be solutions of the adjoint system

$$\frac{\partial X(t,\tau)}{\partial \tau} + \sum_{i=0}^{l} \left(X(t,\tau+ih)A_{11i}(\tau+ih) + Y(t,\tau+ih)A_{21i}(\tau+ih) \right) = 0,$$

$$\tau \leq t, \qquad \tau + ih \neq t - kh, \qquad i = 0,\dots,l, \qquad k = 1,2,\dots, \tag{2.1}$$

$$T_t = [(t-t_0)/h] \quad \text{for} \quad t \geq t_0,$$

$$Y(t,\tau) = \sum_{i=0}^{l} \left(X(t,\tau+ih)A_{12i}(\tau+ih) + Y(t,\tau+ih)A_{22i}(\tau+ih) \right), \qquad \tau \leq t, \tag{2.2}$$

$$X(t,t-kh-0) - X(t,t-kh+0) = \sum_{i=k-l}^{k} Z(t,t-ih)A_{21\,k-i}(t-ih), \tag{2.3}$$

$$Z(t,t-kh) = \sum_{i=k-l}^{k} Z(t,t-ih)A_{22\,k-i}(t-ih), \qquad k = 1,2,\dots,T_t, \tag{2.4}$$

$$Y(t,\tau) = 0, \qquad X(t,\tau) = 0, \qquad Z(t,\tau) = 0, \qquad \tau > t. \tag{2.5}$$

Theorem 2.1. *There exists a unique solution*

$$x(t) = x\left(t; t_0, x_0, \varphi, \psi, \xi, u\right), \qquad y(t) = y\left(t; t_0, x_0, \varphi, \psi, \xi, u\right), \qquad t \geq t_0,$$

of system (1.1) *with the initial conditions* (1.2) *and an admissible control* $u(\tau)$, $\tau \in [t_0, t]$. *It can be computed by the formula*

$$\sum_{i=0}^{l} \int_{t_0}^{t} \left(Y(t,\tau+ih)B_{2i}(\tau+ih) + X(t,\tau+ih)B_{1i}(\tau+ih) \right) u(\tau)d\tau$$

$$+ \sum_{j=0}^{T_t} \sum_{k=j-l}^{j} Z(t,t-kh)B_{2j-k}(t-kh)u(t-jh)$$

$$+ \sum_{i=0}^{l} \int_{t_0-ih}^{t_0} \left(Y(t,\tau+ih)A_{22i}(\tau+ih) + X(t,\tau+ih)A_{12i}(\tau+ih) \right) \psi(\tau)d\tau$$

$$+\sum_{i=0}^{l}\int\limits_{t_0-ih}^{t_0}\left(Y(t,\tau+ih)B_{2i}(\tau+ih)+X(t,\tau+ih)B_{1i}(\tau+ih)\right)\xi(\tau)d\tau$$
$$+\sum_{i=0}^{l}\int\limits_{t_0-ih}^{t_0}\left(X(t,\tau+ih)A_{11i}(\tau+ih)+Y(t,\tau+ih)A_{21i}(\tau+ih)\right)\varphi(\tau)d\tau$$
$$+\sum_{j=T_t+1}^{T_t+l}\sum_{k=j-l}^{T_t}Z(t,t-kh)A_{21j-k}(t-kh)\varphi(t-jh)+X\left(t,t_0-0\right)x_0$$
$$+\sum_{j=T_t+1}^{T_t+l}\sum_{k=j-l}^{T_t}Z(t,t-kh)A_{22j-k}(t-kh)\psi(t-jh)$$
$$+\sum_{j=T_t+1}^{T_t+l}\sum_{k=j-l}^{T_t}Z(t,t-kh)B_{2j-k}(t-kh)\xi(t-jh)$$
$$=\begin{cases} x(t) & \textit{for}\quad t\ge t_0\ \textit{if}\ X(t,t-0)=I_n\ \textit{and}\ Z(t,t)=0\in\mathbb{R}^{n\times m}\\ y(t) & \textit{for}\quad t\ge t_0\ \textit{if}\ X(t,t-0)=A_{210}(t)\in\mathbb{R}^{m\times n}\ \textit{and}\ Z(t,t)=I_m.\end{cases}\tag{2.6}$$

Here and throughout the following, the symbol I_k stands for the identity $k\times k$ matrix.

Proof. The existence and uniqueness of a solution of system (1.1) with the initial conditions (1.2) and a piecewise continuous control can justified by integrating this system by the step method. Let us prove the representation (2.6) on the basis of classical ideas related to adjoint boundary value problems [12].

To be definite, we assume that $t-t_0\ne T_th$ and $T_t>l$. Since the entries of the matrix functions $X(t,\cdot)$ and $Y(t,\cdot)$ and the components of the vector functions $x(\cdot)$, $y(\cdot)$, $u(\cdot)$, $\varphi(\cdot)$, $\psi(\cdot)$, and $\xi(\cdot)$ are piecewise continuous, it follows from the main system (1.1), (1.2) and the adjoint system (2.1)–(2.5) that

$$\int\limits_{t_0}^{t}X(t,\tau)\Bigg(\dot x(\tau)-\sum_{i=0}^{l}A_{11i}(\tau)x(\tau-ih)-\sum_{i=0}^{l}A_{12i}(\tau)y(\tau-ih)$$
$$-\sum_{i=0}^{l}B_{1i}(\tau)u(\tau-ih)\Bigg)d\tau+\int\limits_{t_0}^{t}Y(t,\tau)\Bigg(y(\tau)-\sum_{i=0}^{l}A_{21i}(\tau)x(\tau-ih)$$
$$-\sum_{i=0}^{l}A_{22i}(\tau)y(\tau-ih)-\sum_{i=0}^{l}B_{2i}(\tau)u(\tau-ih)\Bigg)d\tau+\sum_{k=0}^{T_t}Z(t,t-kh)y(t-kh)$$
$$-\sum_{k=0}^{T_t}Z(t,t-kh)\sum_{i=0}^{l}A_{21i}(t-kh)x(t-kh-ih)$$
$$-\sum_{k=0}^{T_t}Z(t,t-kh)\sum_{i=0}^{l}A_{22i}(t-kh)y(t-kh-ih)$$
$$-\sum_{k=0}^{T_t}Z(t,t-kh)\sum_{i=0}^{l}B_{2i}(t-kh)u(t-kh-ih)=0,\qquad t>t_0,$$

or

$$\int\limits_{t_0}^{t}X(t,\tau)\dot x(\tau)d\tau-\int\limits_{t_0}^{t}\sum_{i=0}^{l}\left(X(t,\tau)A_{11i}(\tau)+Y(t,\tau)A_{21i}(\tau)\right)x(\tau-ih)d\tau$$

$$
\begin{aligned}
&+\int_{t_0}^{t} Y(t,\tau)y(\tau)d\tau - \int_{t_0}^{t}\sum_{i=0}^{l}\left(Y(t,\tau)A_{22i}(\tau)+X(t,\tau)A_{12i}(\tau)\right)y(\tau-ih)d\tau \\
&-\int_{t_0}^{t}\sum_{i=0}^{l}\left(Y(t,\tau)B_{2i}(\tau)+X(t,\tau)B_{1i}(\tau)\right)u(\tau-ih)d\tau \\
&+\sum_{k=0}^{T_t} Z(t,t-kh)y(t-kh) - \sum_{k=0}^{T_t} Z(t,t-kh)\sum_{i=0}^{l} A_{21i}(t-kh)x(t-kh-ih) \\
&-\sum_{k=0}^{T_t} Z(t,t-kh)\sum_{i=0}^{l} A_{22i}(t-kh)y(t-kh-ih) \\
&-\sum_{k=0}^{T_t} Z(t,t-kh)\sum_{i=0}^{l} B_{2i}(t-kh)u(t-kh-ih) = 0, \qquad t>t_0. \qquad (2.7)
\end{aligned}
$$

The entries of the matrix function $X(t,\cdot)$ can have only jump discontinuities at the points $\tau = t-kh$, $k=1,2,\ldots,T_t$. Then, by integrating by parts in the first term in (2.7) on each of the intervals $(t-(k+1)h, t-kh)$, $k=0,1,\ldots,T_t$, by performing a shift of the integration variable, and by changing the summation variable according to the formula $i+k=j$, we obtain

$$
\begin{aligned}
&-\sum_{k=0}^{T_t-1}\int_{t-kh-h}^{t-kh}\frac{\partial}{\partial\tau}(X(t,\tau))x(\tau)d\tau - \int_{t_0}^{t-T_th}\frac{\partial}{\partial\tau}(X(t,\tau))x(\tau)d\tau \\
&\quad+\sum_{k=0}^{T_t-1}\left(X(t,t-kh-0)x(t-kh-0) - X(t,t-kh-h+0)x(t-kh-h+0)\right) \\
&\quad+X\left(t,t-T_th-0\right)x\left(t-T_th-0\right) - X\left(t,t_0+0\right)x_0 \\
&\quad-\sum_{i=0}^{l}\int_{t_0-ih}^{t-ih}\left(X(t,\tau+ih)A_{11i}(\tau+ih)+Y(t,\tau+ih)A_{21i}(\tau+ih)\right)x(\tau)d\tau \\
&\quad-\sum_{i=0}^{l}\int_{t_0-ih}^{t-ih}\left(Y(t,\tau+ih)A_{22i}(\tau+ih)+X(t,\tau+ih)A_{12i}(\tau+ih)\right)y(\tau)d\tau \\
&\quad-\sum_{i=0}^{l}\int_{t_0-ih}^{t-ih}\left(Y(t,\tau+ih)B_{2i}(\tau+ih)+X(t,\tau+ih)B_{1i}(\tau+ih)\right)u(\tau)d\tau \\
&\quad+\int_{t_0}^{t} Y(t,\tau)y(\tau)d\tau + Z(t,t)y(t) + \sum_{k=1}^{T_t} Z(t,t-kh)y(t-kh) \\
&\quad-\sum_{k=0}^{T_t} Z(t,t-kh)\sum_{j=k}^{k+l} A_{21j-k}(t-kh)x(t-jh) \\
&\quad-\sum_{k=0}^{T_t} Z(t,t-kh)\sum_{j=k}^{k+l} A_{22j-k}(t-kh)y(t-jh) \\
&\quad-\sum_{k=0}^{T_t} Z(t,t-kh)\sum_{j=k}^{k+l} B_{2j-k}(t-kh)u(t-jh)
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{t_0}^{t} \frac{\partial}{\partial \tau}(X(t,\tau))x(\tau)d\tau + \sum_{k=1}^{T_t}(X(t,t-kh-0) - X(t,t-kh+0))x(t-kh) \\
&\quad + X(t,t-0)x(t) - X\left(t,t_0+0\right)x_0 + \int_{t_0}^{t} Y(t,\tau)y(\tau)d\tau \\
&\quad - \sum_{i=0}^{l} \int_{t_0-ih}^{t-ih} \left(X(t,\tau+ih)A_{11i}(\tau+ih) + Y(t,\tau+ih)A_{21i}(\tau+ih)\right)x(\tau)d\tau \\
&\quad - \sum_{i=0}^{l} \int_{t_0-ih}^{t-ih} \left(Y(t,\tau+ih)A_{22i}(\tau+ih) + X(t,\tau+ih)A_{12i}(\tau+ih)\right)y(\tau)d\tau \\
&\quad - \sum_{i=0}^{l} \int_{t_0-ih}^{t-ih} \left(Y(t,\tau+ih)B_{2i}(\tau+ih) + X(t,\tau+ih)B_{1i}(\tau+ih)\right)u(\tau)d\tau \\
&\quad + Z(t,t)y(t) + \sum_{k=1}^{T_t} Z(t,t-kh)y(t-kh) - \sum_{k=0}^{T_t} Z(t,t-kh)\sum_{j=k}^{k+l} A_{21j-k}(t-kh)x(t-jh) \\
&\quad - \sum_{k=0}^{T_t} Z(t,t-kh)\sum_{j=k}^{k+l} A_{22j-k}(t-kh)y(t-jh) \\
&\quad - \sum_{k=0}^{T_t} Z(t,t-kh)\sum_{j=k}^{k+l} B_{2j-k}(t-kh)u(t-jh), \qquad t > t_0.
\end{aligned}
$$

By changing the order of integration in the last three terms and by taking into account the relation $Z(t,\tau) = 0$, $\tau > t$, we obtain

$$
\begin{aligned}
&-\int_{t_0}^{t} \frac{\partial}{\partial \tau}(X(t,\tau))x(\tau)d\tau + \sum_{k=1}^{T_t}(X(t,t-kh-0) - X(t,t-kh+0))x(t-kh) \\
&\qquad + X(t,t-0)x(t) - X\left(t,t_0+0\right)x_0 + \int_{t_0}^{t} Y(t,\tau)y(\tau)d\tau \\
&\qquad - \sum_{i=0}^{l} \int_{t_0-ih}^{t_0} \left(X(t,\tau+ih)A_{11i}(\tau+ih) + Y(t,\tau+ih)A_{21i}(\tau+ih)\right)\varphi(\tau)d\tau \\
&\qquad - \sum_{i=0}^{l} \int_{t_0}^{t} \left(X(t,\tau+ih)A_{11i}(\tau+ih) + Y(t,\tau+ih)A_{21i}(\tau+ih)\right)x(\tau)d\tau \\
&\qquad - \sum_{i=0}^{l} \int_{t_0-ih}^{t_0} \left(Y(t,\tau+ih)A_{22i}(\tau+ih) + X(t,\tau+ih)A_{12i}(\tau+ih)\right)\psi(\tau)d\tau \\
&\qquad - \sum_{i=0}^{l} \int_{t_0}^{t} \left(Y(t,\tau+ih)A_{22i}(\tau+ih) + X(t,\tau+ih)A_{12i}(\tau+ih)\right)y(\tau)d\tau
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{i=0}^{l} \int_{t_0-ih}^{t_0} \left(Y(t,\tau+ih)B_{2i}(\tau+ih) + X(t,\tau+ih)B_{1i}(\tau+ih)\right)\xi(\tau)d\tau \\
&- \sum_{i=0}^{l} \int_{t_0}^{t} \left(Y(t,\tau+ih)B_{2i}(\tau+ih) + X(t,\tau+ih)B_{1i}(\tau+ih)\right)u(\tau)d\tau \\
&+ Z(t,t)y(t) + \sum_{k=1}^{T_t} Z(t,t-kh)y(t-kh) - Z(t,t)A_{210}(t)x(t) \\
&- \sum_{j=1}^{T_t} \sum_{k=j-l}^{j} Z(t,t-kh)A_{21j-k}(t-kh)x(t-jh) \\
&- \sum_{j=T_t+1}^{T_t+l} \sum_{k=j-l}^{T_t} Z(t,t-kh)A_{21j-k}(t-kh)\varphi(t-jh) \\
&- \sum_{j=0}^{T_t} \sum_{k=j-l}^{j} Z(t,t-kh)A_{22j-k}(t-kh)y(t-jh) \\
&- \sum_{j=T_t+1}^{T_t+l} \sum_{k=j-l}^{T_t} Z(t,t-kh)A_{22j-k}(t-kh)\psi(t-jh) \\
&- \sum_{j=0}^{T_t} \sum_{k=j-l}^{j} Z(t,t-kh)B_{2j-k}(t-kh)u(t-jh) \\
&- \sum_{j=T_t+1}^{T_t+l} \sum_{k=j-l}^{T_t} Z(t,t-kh)B_{2j-k}(t-kh)\xi(t-jh) = 0, \qquad t > t_0.
\end{aligned}
$$

Let us rearrange the terms in the last relation:

$$
\begin{aligned}
&\left(X(t,t-0) - Z(t,t)A_{210}(t)\right)x(t) + Z(t,t)y(t) \\
&\quad - \int_{t_0}^{t} \left(\frac{\partial X(t,\tau)}{\partial \tau} + \sum_{i=0}^{l} \left(X(t,\tau+ih)A_{11i}(\tau+ih) + Y(t,\tau+ih)A_{21i}(\tau+ih)\right) \right) x(\tau)d\tau \\
&\quad + \int_{t_0}^{t} \left(Y(t,\tau) - \sum_{i=0}^{l} \left(Y(t,\tau+ih)A_{22i}(\tau+ih) + X(t,\tau+ih)A_{12i}(\tau+ih)\right) \right) y(\tau)d\tau \\
&\quad + \sum_{k=1}^{T_t} \left(X(t,t-kh-0) - X(t,t-kh+0) - \sum_{i=k-l}^{k} Z(t,t-ih)A_{21k-i}(t-ih) \right) \\
&\quad \times x(t-kh) + \sum_{k=1}^{T_t} \left(Z(t,t-kh) - \sum_{i=k-l}^{k} Z(t,t-ih)A_{22k-i}(t-ih) \right) y(t-kh) \\
&\quad - \sum_{i=0}^{l} \int_{t_0}^{t} \left(Y(t,\tau+ih)B_{2i}(\tau+ih) + X(t,\tau+ih)B_{1i}(\tau+ih)\right)u(\tau)d\tau \\
&\quad - \sum_{j=0}^{T_t} \sum_{k=j-l}^{j} Z(t,t-kh)B_{2j-k}(t-kh)u(t-jh)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{i=0}^{l} \int_{t_0-ih}^{t_0} \left(Y(t,\tau+ih)A_{22i}(\tau+ih) + X(t,\tau+ih)A_{12i}(\tau+ih)\right)\psi(\tau)d\tau \\
&- \sum_{i=0}^{l} \int_{t_0-ih}^{t_0} \left(Y(t,\tau+ih)B_{2i}(\tau+ih) + X(t,\tau+ih)B_{1i}(\tau+ih)\right)\xi(\tau)d\tau \\
&- \sum_{i=0}^{l} \int_{t_0-ih}^{t_0} \left(X(t,\tau+ih)A_{11i}(\tau+ih) + Y(t,\tau+ih)A_{21i}(\tau+ih)\right)\varphi(\tau)d\tau \\
&- X\left(t,t_0+0\right)x_0 - \sum_{j=T_t+1}^{T_t+l} \sum_{k=j-l}^{T_t} Z(t,t-kh)A_{21j-k}(t-kh)\varphi(t-jh) \\
&- \sum_{j=T_t+1}^{T_t+l} \sum_{k=j-l}^{T_t} Z(t,t-kh)A_{22j-k}(t-kh)\psi(t-jh) \\
&- \sum_{j=T_t+1}^{T_t+l} \sum_{k=j-l}^{T_t} Z(t,t-kh)B_{2j-k}(t-kh)\xi(t-jh) = 0, \qquad t > t_0. \qquad (2.8)
\end{aligned}
$$

Since the matrix functions $X(t,\tau)$, $Z(t,\tau)$, and $Y(t,\tau)$ satisfy the adjoint system, it follows that relation (2.8) can be represented in the form

$$
\begin{aligned}
&\left(X(t,t-0) - Z(t,t)A_{210}(t)\right)x(t) + Z(t,t)y(t) \\
&\quad = X\left(t,t_0+0\right)x_0 + \sum_{j=0}^{T_t} \sum_{k=j-l}^{j} Z(t,t-kh)B_{2j-k}(t-kh)u(t-jh) \\
&\qquad + \sum_{i=0}^{l} \int_{t_0}^{t} \left(Y(t,\tau+ih)B_{2i}(\tau+ih) + X(t,\tau+ih)B_{1i}(\tau+ih)\right)u(\tau)d\tau \\
&\qquad + \sum_{i=0}^{l} \int_{t_0-ih}^{t_0} \left(X(t,\tau+ih)A_{11i}(\tau+ih) + Y(t,\tau+ih)A_{21i}(\tau+ih)\right)\varphi(\tau)d\tau \\
&\qquad + \sum_{i=0}^{l} \int_{t_0-ih}^{t_0} \left(Y(t,\tau+ih)A_{22i}(\tau+ih) + X(t,\tau+ih)A_{12i}(\tau+ih)\right)\psi(\tau)d\tau \\
&\qquad + \sum_{i=0}^{l} \int_{t_0-ih}^{t_0} \left(Y(t,\tau+ih)B_{2i}(\tau+ih) + X(t,\tau+ih)B_{1i}(\tau+ih)\right)\xi(\tau)d\tau \\
&\qquad + \sum_{j=T_t+1}^{T_t+l} \sum_{k=j-l}^{T_t} Z(t,t-kh)A_{21j-k}(t-kh)\varphi(t-jh) \\
&\qquad + \sum_{j=T_t+1}^{T_t+l} \sum_{k=j-l}^{T_t} Z(t,t-kh)A_{22j-k}(t-kh)\psi(t-jh) \\
&\qquad + \sum_{j=T_t+1}^{T_t+l} \sum_{k=j-l}^{T_t} Z(t,t-kh)B_{2j-k}(t-kh)\xi(t-jh), \qquad t > t_0, \qquad t - T_t h \neq t_0. \qquad (2.9)
\end{aligned}
$$

Now let $t - T_t h = t_0$. Then in the representation of the integral

$$\int_{t_0}^{t} X(t,\tau)\dot{x}(\tau)d\tau = \sum_{k=0}^{T_t-1} \int_{t-kh-h}^{t-kh} X(t,\tau)\dot{x}(\tau)d\tau + \int_{t_0}^{t-T_t h} X(t,\tau)\dot{x}(\tau)d\tau, \qquad t > t_0,$$

the last term vanishes, and consequently, the jump equation (2.3) in the adjoint system proves to be incomplete. As a result, formula (2.9) remains valid with $X(t,t_0+0)$ replaced by $X(t,t_0-0)$; i.e., the term $-X(t,t_0+0)\,x(t_0)$ is replaced by $-X(t,t_0-0)\,x(t_0)$, which is valid for $t - T_t h \neq t_0$ as well, since in the last case, the matrix function $X(t,\tau)$ is continuous for $\tau = t_0$:

$$X(t,t_0+0) = X(t,t_0-0).$$

Therefore, relation (2.9), together with the boundary conditions for the adjoint system, implies that the representation (2.6) is valid for arbitrary t, $T_t > l$. In a similar way, one can justify relation (2.9) for $T_t \leq l$.

A straightforward verification shows that the representation (2.6) also remains valid for $t = t_0$. The proof of Theorem 2.1 is complete.

Remark 2.1. Note that if $x(t)$, $t \geq t_0$, then $Z(t,t-kh) = 0$, $k = 0,1,\ldots,T_t$, and, as follows from (2.3), $X(t,\tau)$ is a continuous function for $\tau \leq t$. Therefore, terms containing the function $Z(t,\cdot)$ in (2.6) can be omitted in this case.

Remark 2.2. In the Cauchy formula (2.6), one can set $X(t,t_0-0) = X(t,t_0)$ if the matrix function $X(t,\tau)$ is assumed to be left continuous with respect to the second argument.

3. REPRESENTATION OF SOLUTIONS OF STATIONARY HYBRID DIFFERENCE-DIFFERENTIAL SYSTEMS

Theorem 3.1. *The solution of system* (1.1)–(1.3) *with an admissible control* $u(\tau)$, $\tau \in [0,t]$, *exists, is unique, and can be represented in the form*

$$\begin{aligned} x(t) = &\int_0^t \sum_{j=0}^{l} \left(X_x^*(t-\tau-jh)B_{1j} + Y_x^*(t-\tau-jh)B_{2j}\right)u(\tau)d\tau \\ &+ x(t;0,x_0,\varphi,\psi,\xi,0), \qquad t \geq 0, \end{aligned} \tag{3.1}$$

with the initial conditions $X_x^*(0) = X^*(0) = I_n$ *and* $Z_x^*(0) = Z^*(0) = 0 \in \mathbb{R}^{n\times m}$ *and*

$$\begin{aligned} y(t) = &\int_0^t \sum_{j=0}^{l} \left(X_y^*(t-\tau-jh)B_{1j} + Y_y^*(t-\tau-jh)B_{2j}\right)u(\tau)d\tau \\ &+ \sum_{j=0}^{T_t} \sum_{k=j-l}^{j} Z_y^*(kh)B_{2j-k}u(t-jh) + y(t;0,x_0,\varphi,\psi,\xi,0), \qquad t \geq 0, \end{aligned} \tag{3.2}$$

with the initial conditions $X_y^*(0) = X^*(0) = A_{210} \in \mathbb{R}^{m\times n}$ *and* $Z_y^*(0) = Z^*(0) = I_m$. *Here the* n*-vector function* $x(t;0,x_0,\xi,\psi,\varphi,0)$ *and the* m*-vector function* $y(t;0,x_0,\xi,\psi,\varphi,0)$, $\tau \in [-lh,0]$, *depend only on the initial data and have the form*

$$\begin{aligned} x(t;0,x_0,\varphi,\psi,\xi,0) = {} & \sum_{j=0}^{l} \int_{-jh}^{0} \left(X_x^*(t-\tau-jh)A_{12j} + Y_x^*(t-\tau-jh)A_{22j}\right)\psi(\tau)d\tau \\ & + \sum_{j=0}^{l} \int_{-jh}^{0} \left(X_x^*(t-\tau-jh)B_{1j} + Y_x^*(t-\tau-jh)B_{2j}\right)\xi(\tau)d\tau \\ & + \sum_{j=0}^{l} \int_{-jh}^{0} \left(X_x^*(t-\tau-jh)A_{11j} + Y_x^*(t-\tau-jh)A_{21j}\right)\varphi(\tau)d\tau \\ & + X_x^*(t)x_0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} y(t;0,x_0,\varphi,\psi,\xi,0) = {} & \sum_{j=T_t+1}^{T_t+l} \sum_{k=j-l}^{T_t} Z_y^*(kh)\left(A_{21j-k}\varphi(t-jh) + A_{22j-k}\psi(t-jh)\right. \\ & \left. + B_{2j-k}\xi(t-jh)\right) + \sum_{j=0}^{l} \int_{-jh}^{0} \Big(X_y^*(t-\tau-jh)A_{11j} \\ & + Y_y^*(t-\tau-jh)A_{21j}\Big)\varphi(\tau)d\tau + X_y^*(t)x_0 \\ & + \sum_{j=0}^{l} \int_{-jh}^{0} \Big(X_y^*(t-\tau-jh)A_{12j} + Y_y^*(t-\tau-jh)A_{22j}\Big)\psi(\tau)d\tau \\ & + \sum_{j=0}^{l} \int_{-jh}^{0} \Big(X_y^*(t-\tau-jh)B_{1j} + Y_y^*(t-\tau-jh)B_{2j}\Big)\xi(\tau)d\tau, \end{aligned} \tag{3.4}$$

where the matrix functions $X^*(\cdot)$, $Z^*(\cdot)$, *and* $Y^*(\cdot)$ *are solution of the adjoint system*

$$-\frac{dX^*(t)}{dt} + \sum_{j=0}^{l}\left(X^*(t-jh)A_{11j} + Y^*(t-jh)A_{21j}\right) = 0, \quad t \ge 0, \quad t \ne kh, \tag{3.5}$$

$$Y^*(t) = \sum_{j=0}^{l}\left(X^*(t-jh)A_{12j} + Y^*(t-jh)A_{22j}\right), \qquad t \ge 0, \tag{3.6}$$

$$X^*(kh) - X^*(kh-0) = \sum_{j=k-l}^{k} Z^*(jh)A_{21k-j}, \tag{3.7}$$

$$Z^*(kh) = \sum_{j=k-l}^{k-1} Z^*(jh)A_{22k-j}, \tag{3.8}$$

for $t \ge 0$, $k = 1, \dots, T_t$; *moreover,*

$$Y^*(t) = 0, \qquad X^*(t) = 0, \qquad Z^*(t) = 0, \qquad t < 0. \tag{3.9}$$

Remark 3.1. At points of discontinuity $t = kh$, $k = 0, 1, \dots, T_t$, of the matrix functions $X^*(t)$, the right derivatives are considered in Eq. (3.5), and the matrix functions themselves are assumed to be right continuous.

Theorem 3.1 follows from Theorem 2.1 in view of the fact that, in the stationary case, the solution of the adjoint system (2.1)–(2.5) is given by the matrix functions $X(t,\tau) = X^*(t-\tau)$, $Y(t,\tau) = Y^*(t-\tau)$, and $Z(t,\tau) = Z^*(t-\tau)$, where $X^*(\cdot)$, $Z^*(\cdot)$, and $Y^*(\cdot)$ are solutions of system (3.5)–(3.9) with the ordinary direction of time $t \geq t_0 = 0$.

4. EXAMPLES

4.1. Consider the hybrid difference-differential system

$$\dot{x}(t) = x(t) + y(t), \qquad y(t) = x(t) + y(t-1), \qquad t \geq 0, \tag{4.1}$$

where $x(t), y(t) \in \mathbb{R}$ and the initial conditions are given in the form

$$x(+0) = x(0) = 0, \qquad y(\tau) = \psi(\tau) = \begin{cases} 0 & \text{for} \quad \tau \in (-1,0) \\ 1 & \text{for} \quad \tau = -1. \end{cases}$$

The adjoint system (3.5)–(3.9) acquires the form

$$\dot{x}^*(t) = x^*(t) + y^*(t), \qquad t \geq 0, \qquad t \neq k, \tag{4.2}$$
$$y^*(t) = x^*(t) + y^*(t-1), \qquad t \geq 0, \tag{4.3}$$
$$y^*(t) = 0, \qquad t < 0, \tag{4.4}$$
$$x^*(k) - x^*(k-0) = z^*(k), \tag{4.5}$$
$$z^*(k) = z^*(k-1), \qquad k = 1, 2, \ldots, T_t. \tag{4.6}$$

Then, on the basis of the representations (3.3) and (3.4) and the initial conditions, we obtain the solution of system (4.1) in the form

$$x(t) \equiv 0 \quad \text{for} \quad t \geq 0 \quad \left[\text{here} \quad x_x^*(0) = 1, \quad z_x^*(0) = 0\right],$$
$$y(t) = z_y^*(T_t)\,\psi(t - T_t - 1) = \begin{cases} 0 & \text{for} \quad t \neq k \\ 1 & \text{for} \quad t = k,\ k = 0, 1, \ldots, T_t \end{cases}$$
$$\left[\text{here} \quad x_y^*(0) = 1, \quad z_y^*(k) = 1, \quad k = 0, 1, \ldots\right].$$

Note that such a solution cannot be obtained by the standard application of the Laplace transform to system (4.1).

4.2. Let us now illustrate the scheme of using the representations (3.3) and (3.4) by the following initial value problem for system (4.1):

$$x(0) = 1, \qquad y(\tau) = \psi(\tau) = -1, \qquad \tau \in [-1, 0).$$

To be definite, we find the corresponding solution $x(t)$, $y(t)$, where $1 < t < 2$.

To find $x(t)$, we solve the adjoint system (4.2)–(4.6) by the step method for $x_x^*(0) = 1$ and $z_x^*(0) = 0$ and use the representation (3.3) at each step. From (4.6) and (4.5), we successively find $z_x^*(k) = 0$ and $x_x^*(k) = x_x^*(k-0)$, $k = 1, 2, \ldots, T_t$. We have $x_x^*(t) = y_x^*(t) = e^{2t}$ and $x_x^*(1) = e^2$ for $t \in [0,1)$. It follows from the representation (3.3) that ($T_t = 0$)

$$x(t) = -\int_{-1}^{0} y_x^*(t - \tau - 1)\,d\tau + x_x^*(t) = \int_{t}^{t-1} y_x^*(s)\,ds + e^{2t}$$
$$= -\int_{t-1}^{0} y_x^*(s)\,ds - \int_{0}^{t} y_x^*(s)\,ds + e^{2t} = -\int_{0}^{t} e^{2s}\,ds + e^{2t} = \frac{1}{2} + \frac{1}{2}e^{2t}.$$

If $t \in [1,2)$, then $y_x^*(t) = x_x^*(t) + e^{2(t-1)}$ and

$$x_x^*(t) = e^{2(t-1)}e^2 + \int_{1}^{t} e^{2(t-\tau)}e^{2(\tau-1)}\,d\tau = e^{2(t-1)}\left(e^2 + t - 1\right),$$
$$y_x^*(t) = \left(t + e^2\right)e^{2(t-1)}, \qquad x_x^*(2) = e^2\left(1 + e^2\right).$$

From the representation (3.3), we obtain ($T_t = 1$)

$$x(t) = -\int\limits_{-1}^{0} y_x^*(t-\tau-1)d\tau + x_x^*(t) = \int\limits_{t}^{t-1} y_x^*(s)ds + e^{2(t-1)}\left(e^2+t-1\right)$$
$$= -\int\limits_{t-1}^{1} y_x^*(s)ds - \int\limits_{1}^{t} y_x^*(s)ds + e^{2(t-1)}\left(e^2+t-1\right)$$
$$= -\int\limits_{t-1}^{1} e^{2s}ds - \int\limits_{1}^{t}\left(s+e^2\right)e^{2(s-1)}ds + e^{2(t-1)}\left(e^2+t-1\right)$$
$$= -\frac{1}{2}\left(e^2-e^{2(t-1)}\right) - \frac{1}{2}\left(e^2+t\right)e^{2(t-1)} + \frac{1}{2}\left(e^2+1\right) + \frac{1}{4}\left(e^{2(t-1)}-1\right)$$
$$+ e^{2(t-1)}\left(e^2+t-1\right) = e^{2(t-1)}\left(\frac{1}{2}e^2+\frac{1}{2}t-\frac{1}{4}\right)+\frac{1}{4}.$$

To find $y(t)$, we solve the adjoint system (4.2)–(4.6) by the step method for $x_y^*(0) = 1$ and $z_y^*(0) = 1$ and use the representation (3.4) at each step. From (4.6) and (4.5), we successively find $z_y^*(k) = 1$ and $x_y^*(k) = 1 + x_y^*(k-0)$, $k = 1, 2, \ldots, T_t$. If $t \in [0,1)$, then $x_y^*(t) = y_y^*(t) = e^{2t}$ and $x_y^*(1) = 1 + e^2$. From the representation (3.4), we obtain ($T_t = 0$)

$$y(t) = -\int\limits_{-1}^{0} y_y^*(t-\tau-1)d\tau - z_y^*\left(T_t\right) + x_y^*(t) = \int\limits_{t}^{t-1} y_y^*(s)ds - 1 + e^{2t}$$
$$= -\int\limits_{t-1}^{0} y_y^*(s)ds - \int\limits_{0}^{t} y_y^*(s)ds - 1 + e^{2t} = \frac{1}{2}e^{2t} - \frac{1}{2}.$$

We have $x_y^*(t) = e^{2(t-1)}\left(t+e^2\right)$ and $y_y^*(t) = e^{2(t-1)}\left(t+e^2+1\right)$ for $t \in [1,2)$.

From the representation (3.4), we obtain ($T_t = 1$)

$$y(t) = -\int\limits_{-1}^{0} y_y^*(t-\tau-1)d\tau - z_y^*\left(T_t\right) + x_y^*(t) = \int\limits_{t}^{t-1} y_y^*(s)ds - 1 + e^{2(t-1)}\left(t+e^2\right)$$
$$= -\int\limits_{t-1}^{1} y_y^*(s)ds - \int\limits_{1}^{t} y_y^*(s)ds - 1 + e^{2(t-1)}\left(t+e^2\right)$$
$$= -\int\limits_{t-1}^{1} e^{2s}ds - \int\limits_{1}^{t} e^{2(s-1)}\left(s+e^2+1\right)ds - 1 + e^{2(t-1)}\left(t+e^2\right)$$
$$= e^{2(t-1)}\left(\frac{1}{2}t+\frac{1}{2}e^2+\frac{1}{4}\right)-\frac{1}{4}.$$

5. IMPLICIT CRITERION FOR THE H-t_1-CONTROLLABILITY OF NONSTATIONARY HYBRID DIFFERENCE-DIFFERENTIAL SYSTEMS

Consider the problem on the relative controllability with a projection H for linear hybrid difference-differential systems with numerous delays. The investigation of this problem is performed on the basis of properties of the attainability set of nonstationary and stationary systems, which are defined on the basis of the corresponding representations of the solutions (2.6) and (3.3), (3.4). In some special cases of hybrid difference-differential systems, the relative controllability problem was considered in [3, 17].

Definition 5.1. System (1.1), (1.2) is said to be *relatively H-t_1-controllable* for $t_1 > t_0$ if for arbitrary vectors $x_0 \in \mathbb{R}^n$ and $\begin{vmatrix} x_1 \\ y_1 \end{vmatrix} \in \mathbb{R}^{n+m}$ and arbitrary admissible initial data $\psi(\tau)$, $\varphi(\tau)$, and $\xi(\tau)$, $\tau \in [t_0 - lh, t_0]$, there exists an admissible control $u(\cdot)$ such that the corresponding solution of system (1.1), (1.2) has the property

$$H \begin{bmatrix} x(t_1) \\ y(t_1) \end{bmatrix} = H \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

If $H = I_{n+m}$, then the system is said to be *relatively t_1-controllable*; for $H = [I_n\ 0]$, it is said to be *relatively t_1-controllable with respect to x*, and for $H = [0\ I_m]$, it is said to be *relatively t_1-controllable with respect to y*.

By $C\Sigma_{(t_0,t_1)}$ we denote the set of continuity points of the system parameters, that is, points τ, $\tau \in (t_0, t_1)$, at which all matrix functions $A_{jki}(\tau)$, $j,k = 1,2$, $B_{1i}(\tau)$, $B_{2i}(\tau)$, $X(t_1,\tau)$, $Z(t_1,\tau)$, and $Y(t_1,\tau)$ are continuous.

By taking into account the separation of motions in the system under the action of controls and the initial data, in the analysis of relative controllability, one can assume without loss of generality that initial data are zero.

The H-attainability set of system (1.1) with zero initial data at time t_1 can be written out on the basis of the solution representation (2.6) in the form

$$\begin{aligned} K(t_1) = \Bigg\{ \gamma \in \mathbb{R}^s :& \\ \gamma =& \int_{t_0}^{t_1} H \sum_{i=0}^{l} \begin{bmatrix} X_x(t_1, \tau+ih) & Y_x(t_1,\tau+ih) \\ X_y(t_1,\tau+ih) & Y_y(t_1,\tau+ih) \end{bmatrix} \begin{bmatrix} B_{1i}(\tau+ih) \\ B_{2i}(\tau+ih) \end{bmatrix} u(\tau) d\tau \\ &+ H \sum_{k=0}^{T_{t_1}} \sum_{i=k-l}^{k} \begin{bmatrix} 0 \\ Z_y(t_1, t_1-kh) \end{bmatrix} B_{2k-i}(t_1 - ih)\, u(t_1-kh) \\ &\forall u(\cdot) \in PC([t_0,t_1], \mathbb{R}^r) \Bigg\}, \end{aligned}$$

where $X_x(t,\tau)$, $Y_x(t,\tau)$, and $Z_x(t,\tau)$ are the solutions of the adjoint system (2.1)–(2.5) with the initial conditions $X_x(t,t-0) = X(t,t-0) = I_n$ and $Z_x(t,t) = Z(t,t) = 0 \in \mathbb{R}^{n\times m}$, and $X_y(t,\tau)$, $Y_y(t,\tau)$, and $Z_y(t,\tau)$ are the solutions of the adjoint system (2.1)–(2.5) with the initial conditions $X_x(t,t-0) = X(t,t-0) = A_{210}(t) \in \mathbb{R}^{m\times n}$ and $Z_x(t,t) = Z(t,t) = I_m$.

Let $K_0 = \{H\mu : \ \forall \mu \in \mathbb{R}^d\}$ be the linear span of the columns of the matrix $H \in \mathbb{R}^{k\times d}$, $d = n+m$; then the relative H-t_1-controllability is equivalent to the inclusion $K(t_1) \supset K_0$ or $(K(t_1))^\perp \subset K_0^\perp$ for the orthogonal complements, which, in turn, is equivalent to the following assertion.

Theorem 5.1. *System* (1.1), (1.2) *is relatively H-t_1-controllable if and only if the relation $g'H = 0$ is valid for any vector $g \in \mathbb{R}^s$ such that*

$$\begin{aligned} &g'H \sum_{i=0}^{l} \begin{bmatrix} X_x(t_1,\tau+ih) & Y_x(t_1,\tau+ih) \\ X_y(t_1,\tau+ih) & Y_y(t_1,\tau+ih) \end{bmatrix} \begin{bmatrix} B_{1i}(\tau+ih) \\ B_{2i}(\tau+ih) \end{bmatrix} = 0, \\ &(\tau+ih) \in C\Sigma_{(t_0,t_1)}, \qquad i = 0,\dots,l, \\ &g'H \sum_{i=k-l}^{k} \begin{bmatrix} 0 \\ Z_y(t_1, t_1-ih) \end{bmatrix} B_{2\,k-i}(t_1-ih) = 0, \qquad k = 0,1,\dots,T_{t_1}. \end{aligned}$$

Proof. We have

$$\Big((K(t_1))^{\perp} \subset K_0^{\perp}\Big) \ \Leftrightarrow \ (\forall g \in \mathbb{R}^s \ g'K(t_1) = 0 \ \Rightarrow \ g'K_0 = 0)$$

$$\Leftrightarrow \Bigg(\forall g \in \mathbb{R}^s$$

$$g'H \int\limits_{t_0}^{t_1} \sum_{i=0}^{l} \begin{bmatrix} X_x(t_1, \tau + ih) & Y_x(t_1, \tau + ih) \\ X_y(t_1, \tau + ih) & Y_y(t_1, \tau + ih) \end{bmatrix} \begin{bmatrix} B_{1i}(\tau + ih) \\ B_{2i}(\tau + ih) \end{bmatrix} u(\tau) d\tau$$

$$+ g'H \sum_{k=0}^{T_{t_1}} \sum_{i=k-l}^{k} \begin{bmatrix} 0 \\ Z_y(t_1, t_1 - ih) \end{bmatrix} B_{2k-i}(t_1 - ih)\, u(t_1 - ih) = 0$$

$$\forall u(\cdot) \in PC([t_0, t_1], \mathbb{R}^r) \ \Rightarrow \ g'H = 0\Bigg)$$

$$\Leftrightarrow \Bigg(\forall g \in \mathbb{R}^p$$

$$\int\limits_{t_0}^{t_1} \left\| g'H \sum_{i=0}^{l} \begin{bmatrix} X_x(t_1, \tau + ih) & Y_x(t_1, \tau + ih) \\ X_y(t_1, \tau + ih) & Y_y(t_1, \tau + ih) \end{bmatrix} \begin{bmatrix} B_{1i}(\tau + ih) \\ B_{2i}(\tau + ih) \end{bmatrix} \right\|^2 d\tau$$

$$+ \sum_{k=0}^{T_{t_1}} \left\| g'H \sum_{i=k-l}^{k} \begin{bmatrix} 0 \\ Z_y(t_1, t_1 - ih) \end{bmatrix} B_{2k-i}(t_1 - ih) \right\|^2 = 0 \ \Rightarrow \ g'H = 0\Bigg)$$

$$\Leftrightarrow \Bigg(\forall g \in \mathbb{R}^p \quad g'H \sum_{i=0}^{l} \begin{bmatrix} X_x(t_1, \tau + ih) & Y_x(t_1, \tau + ih) \\ X_y(t_1, \tau + ih) & Y_y(t_1, \tau + ih) \end{bmatrix} \begin{bmatrix} B_{1i}(\tau + ih) \\ B_{2i}(\tau + ih) \end{bmatrix} = 0,$$

$$(\tau + ih) \in C\Sigma_{(t_0, t_1)}, \qquad i = 0, \ldots, l,$$

$$g'H \sum_{i=k-l}^{k} \begin{bmatrix} 0 \\ Z_y(t_1, t_1 - ih) \end{bmatrix} B_{2k-i}(t_1 - ih) = 0, \quad k = 0, 1, \ldots, T_{t_1}, \quad g'H = 0\Bigg),$$

and the proof of the theorem is complete.

Remark 5.1. In the proof of the necessity of the assertion of Theorem 5.1, the control function has been chosen in the form

$$u(\tau) = \left(g'H \sum_{i=0}^{l} \begin{bmatrix} X_x(t_1, \tau + ih) & Y_x(t_1, \tau + ih) \\ X_y(t_1, \tau + ih) & Y_y(t_1, \tau + ih) \end{bmatrix} \begin{bmatrix} B_{1i}(\tau + ih) \\ B_{2i}(\tau + ih) \end{bmatrix}\right)',$$

$$(\tau + ih) \in C\Sigma_{(t_0, t_1)}, \qquad i = 0, \ldots, l,$$

$$u(t_1 - kh) = \sum_{i=k-l}^{k} \left(g'H \begin{bmatrix} 0 \\ Z_y(t_1, t_1 - ih) \end{bmatrix} B_{2k-i}(t_1 - ih)\right)', \qquad k = 0, 1, \ldots, T_{t_1}.$$

An implicit criterion for the H-t_1-controllability of stationary hybrid systems is a straightforward consequence of Theorem 5.1.

Theorem 5.2. *System* (1.1)–(1.3) *is relatively* H-t_1*-controllable if and only if*

$$\operatorname{Span}\Bigg\{ H\sum_{i=0}^{l}\begin{bmatrix} X_x(t_1-\tau-ih) & Y_x(t_1-\tau-ih) \\ X_y(t_1-\tau-ih) & Y_y(t_1-\tau-ih)\end{bmatrix}\begin{bmatrix} B_{1i} \\ B_{2i}\end{bmatrix}, \quad (t_1-\tau-ih)\in C\Sigma_{(0,t_1)}, \quad i=0,\ldots,l, \quad H\sum_{i=k-l}^{k}\begin{bmatrix} 0 \\ Z_y(ih)\end{bmatrix} B_{2k-i}=0, \quad k=0,1,\ldots,T_{t_1},H \Bigg\}$$

$$=\operatorname{Span}\Bigg\{ H\sum_{i=0}^{l}\begin{bmatrix} X_x(t_1-\tau-ih) & Y_x(t_1-\tau-ih) \\ X_y(t_1-\tau-ih) & Y_y(t_1-\tau-ih)\end{bmatrix}\begin{bmatrix} B_{1i} \\ B_{2i}\end{bmatrix}, \quad (t_1-\tau-ih)\in C\Sigma_{(0,t_1)}, \quad i=0,\ldots,l, \quad H\sum_{i=k-l}^{k}\begin{bmatrix} 0 \\ Z_y(ih)\end{bmatrix} B_{2k-i}=0, \quad k=0,1,\ldots,T_{t_1} \Bigg\},$$

where the symbol $\operatorname{Span}\{Q_1(\tau),Q_2(\tau),\ldots,Q_k(\tau),\ \tau\in(t_0,t_1)\}$ *stands for linear span of the columns of the matrices* $Q_1(\tau),Q_2(\tau),\ldots,Q_k(\tau)$ *for* $\tau\in(t_0,t_1)$.

CONCLUSION

For hybrid difference-differential systems with retarded argument, one can pose the Cauchy problem in a special way, which always has a unique solution on the interval $[t_0,t]$. An adjoint system, which is characterized by the presence of jump equations, is constructed for a time-dependent system. For the adjoint system with various initial conditions, we have obtained an integral representation of the solution in the form of the variation-of-constants formula for the continuous and piecewise continuous components of the vector of phase coordinates, which generalizes earlier obtained results for ordinary systems and systems with retarded argument. The results can be used for the analysis of the relative controllability of algebraic-differential delay systems and can be refined in the stationary case. Moreover, for stationary systems, one can obtain representations of solutions of hybrid difference-differential systems in the form of series [14] in solutions of their determining equations, which generalizes the representation of solutions of ordinary systems on the basis of the expansion of a matrix exponential. This result permits one to extract finitely many generators in the linear span of the solutions and hence to obtain an efficient parametric criterion for the relative controllability of stationary hybrid difference-differential systems.

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