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CONTROL IN DETERMINISTIC SYSTEMS

Linear Stationary Differential–Algebraic Systems: I. Solution Representation

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Abstract—Linear stationary differential–algebraic control systems are studied. An exponential estimate for the solution growth is proved for these systems. The obtained result allows one to apply the Laplace transform for investigation of stationary systems and, as a consequence, to obtain analytical representations of solutions in the form of series in powers of the solutions to the determining equations, which generalizes similar formulas (by expanding the matrix exponential into a series) for ordinary systems.

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INTRODUCTION

In the investigation of real physical processes, algebraic (functional) dependences occur along with dynamical (differential) ones. Such processes are described by differential–algebraic systems [1] containing differential and algebraic equations. They are classified as hybrid systems [2–5]. It should be admitted, however, that the term "hybrid systems" is overloaded. At present, especially in the English-language literature, the term "hybrid" is used mainly in relation to discrete–continuous systems or systems containing logical variables [6–9]. The hybrid character means, generally speaking, a natural inhomogeneity of the process under study or an inhomogeneity of methods for its study. The term "hybrid" is applied to systems describing processes or objects with essentially different characteristics, for example, whose main dynamics contains continuous and discrete variables (signals), deterministic and random values or impacts, and so on, which finally determines the character (nature) of hybrid systems. There are many examples of hybrid systems [2–9].

Below, mixed delay differential–algebraic systems are considered. In particular, some standard types of linear discrete–continuous systems can be reduced to such systems. They can be classified as hybrid differential–difference (HDD) systems or completely regular delay differential–algebraic systems [10] which represent, in turn, a particular case of descriptor (singular) systems with aftereffect. The results of this work were announced earlier in [11]. Their totality can be classified as a theoretical background of analysis and synthesis of real control systems.

1. STATEMENT OF THE PROBLEM

Consider a stationary HDD system of the form

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{l} A_{11i}x(t-ih) + \sum_{i=0}^{l} A_{12i}y(t-ih) + \sum_{i=0}^{l} B_{1i}u(t-ih), \\ y(t) &= \sum_{i=0}^{l} A_{21i}x(t-ih) + \sum_{i=0}^{l} A_{22i}y(t-ih) + \sum_{i=0}^{l} B_{2i}u(t-ih), \quad t \ge 0, \end{aligned} \tag{1.1}$$

with the initial conditions

$$\begin{aligned} &x(+0) = x(0) = x_0 \in \mathbb{R}^n, \quad x(t) = \varphi(t), \\ &y(t) = \psi(t), \quad u(t) = \xi(t), \quad t \in (-\infty, 0); \\ &\varphi(\tau) = 0, \quad \psi(\tau) = 0, \quad \xi(\tau) = 0, \quad \tau \in (-\infty, -lh). \end{aligned} \tag{1.2}$$

Here, $l \in \mathbb{N}$, $A_{220} \equiv 0$, $A_{11i} \in \mathbb{R}^{n \times n}$, $A_{12i} \in \mathbb{R}^{n \times m}$, $A_{21i} \in \mathbb{R}^{m \times n}$, $A_{22i} \in \mathbb{R}^{m \times m}$, $B_{1i} \in \mathbb{R}^{n \times r}$, $B_{2i} \in \mathbb{R}^{m \times r}$ ($i = 0, 1, \ldots, l$); $\psi(\cdot) \in PC((-\infty, 0], \mathbb{R}^m)$, $\varphi(\cdot) \in PC((-\infty, 0], \mathbb{R}^n)$, $\xi(\cdot) \in PC((-\infty, 0], \mathbb{R}^r)$, where $PC(\Delta, \mathbb{R}^n)$ is the set of Δ n-vector functions piecewise-continuous on the set Δ. The vector functions φ, ψ, ξ and the n-vector x_0 are called admissible initial conditions, and the vector function $u(\cdot) \in PC([-lh, +\infty), \mathbb{R}^r)$ is referred to as an admissible control.

Definition 1. The solution $x(t) = x(t; x_0, \varphi, \psi, \xi, u)$, $y(t) = y(t; x_0, \varphi, \psi, \xi, u)$, $t \geq 0$, to system (1.1) corresponding to initial conditions (1.2) and an admissible control $u = u(t)$, $t \geq 0$, is a pair of arbitrary vector functions $x(t)$ and $y(t)$, $t \geq 0$, satisfying the first equation of the system for $t \geq 0$, $t \neq kh$, $k = 0, 1, \ldots$, and the second equation of the system for $t \geq 0$ under the assumption that $x(\cdot)$ is a continuous piecewise-smooth vector function and $y(\cdot)$ is piecewise continuous on the interval $[0, +\infty)$.

Below, for systems of the form (1.1), (1.2), representations of their solutions in the form of series in powers of solutions to the determining equations for continuous and piecewise-continuous controls are obtained.

2. ALGEBRAIC PROPERTIES OF SOLUTIONS TO DETERMINING EQUATIONS

Similarly to [3, 12], the determining equation for HDD system (1.1), (1.2) is introduced as

$$\begin{aligned} X_{k+1}(t) &= \sum_{i=0}^{l}(A_{11i}X_k(t-ih) \\ &+ A_{12i}Y_k(t-ih) + B_{1i}U_k(t-ih)), \\ Y_k(t) &= \sum_{i=0}^{l}(A_{21i}X_k(t-ih) \\ &+ A_{22i}Y_k(t-ih) + B_{2i}U_k(t-ih)), \\ &t \geq 0, \quad k = -1, 0, 1, \ldots, \end{aligned} \tag{2.1}$$

with the initial conditions

$$X_k(t) = 0, \quad Y_k(t) = 0, \text{ if } k < 0 \text{ or } t < 0;$$
$$U_k(t) = 0, \text{ if } k^2 + t^2 \neq 0, \quad U_0(0) = I_r.$$

Lemma 1. The following identities are satisfied:

$$\begin{aligned} &\left(\sum_{i=0}^{l}A_{11i}\omega^i + \sum_{i=0}^{l}A_{12i}\omega^i\left(I_m - \sum_{i=0}^{l}A_{22i}\omega^i\right)^{-1} \right.\\ &\left.\times \sum_{i=0}^{l}A_{21i}\omega^i\right)^k\left(\sum_{i=0}^{l}B_{1i}\omega^i + \sum_{i=0}^{l}A_{12i}\omega^i\right. \\ &\left.\times\left(I_m - \sum_{i=0}^{l}A_{22i}\omega^i\right)^{-1}\sum_{i=0}^{l}B_{2i}\omega^i\right) \equiv \sum_{j=0}^{+\infty}X_{k+1}(jh)\omega^j; \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\left(I_m - \sum_{i=0}^{l}A_{22i}\omega^i\right)^{-1}\sum_{i=0}^{l}A_{21i}\omega^i\left(\sum_{i=0}^{l}A_{11i}\omega^i\right. \\ &\left. + \sum_{i=0}^{l}A_{12i}\omega^i\left(I_m - \sum_{i=0}^{l}A_{22i}\omega^i\right)^{-1}\sum_{i=0}^{l}A_{21i}\omega^i\right)^k \\ &\times\left(\sum_{i=0}^{l}B_{1i}\omega^i + \sum_{i=0}^{l}A_{12i}\omega^i\left(I_m - \sum_{i=0}^{l}A_{22i}\omega^i\right)^{-1}\sum_{i=0}^{l}B_{2i}\omega^i\right) \\ &\equiv \sum_{j=0}^{+\infty}Y_{k+1}(jh)\omega^j; \end{aligned} \tag{2.3}$$

$$\left(I_m - \sum_{i=0}^{l}A_{22i}\omega^i\right)^{-1}\sum_{i=0}^{l}B_{2i}\omega^i \equiv \sum_{j=0}^{+\infty}Y_0(jh)\omega^j, \tag{2.4}$$

where $k = 0, 1, \ldots$; $|\omega| \leq \omega_1$, ω_1 is a sufficiently small positive number.

The proof of the lemma can be performed similarly to [13, 14] using the method of mathematical induction.

Remark 1. The convergence of the matrix series in Lemma 1 is determined by the convergence of the corresponding matrix geometric series in the right-hand side of the obtained relations, which is always valid for sufficiently small absolute values of ω, in particular, for

$$\left\|\sum_{i=0}^{l}A_{22i}\omega^i\right\| < 1$$

or

$$|\omega| \leq \omega_1 = \frac{1}{l\max\limits_{1\leq i\leq l}\|A_{22i}\|}.$$

3. GROWTH ESTIMATE FOR SOLUTIONS TO STATIONARY HDD SYSTEMS

Assume that

$$M_1 = \sup_{t\in[-lh,0]}\|\varphi(t)\|, \quad M_2 = \sup_{t\in[-lh,0]}\|\psi(t)\|,$$
$$M_3 = \sup_{t\in[-lh,0]}\|\xi(t)\|.$$

In order to obtain a growth estimate for the solutions to stationary system (1.1), the following lemma is necessary.

Lemma 2. [15] If $c \geq 0$, $u(t) \geq 0$, $\nu(t) \geq 0$ and the relation

$$u(t) \leq c + \int_a^t u(\tau)\nu(\tau)d\tau$$

is satisfied for $t > 0$, then the function $u(\cdot)$ satisfies the inequality

$$u(t) \le c e^{\int_a^t \nu(\tau)d\tau} \quad \text{if } t > 0.$$

Theorem 1. For each solution to system (1.1) corresponding to admissible initial conditions (1.2) and admissible control $u(\cdot)$ whose growth rate does not exceed an exponential one, i.e., $\|u(t)\| \le Me^{\sigma t}$, $t \ge 0$ (M, σ are positive constants), positive numbers L and α can be found such that $\|x(t)\| \le Le^{\alpha t}$, $\|y(t)\| \le Le^{\alpha t}$, $t \ge 0$, where L and α may depend only on M_1, M_2, M_3, M, σ, and the parameters of the system.

Proof. Introduce the notation

$$D = \max\{1, M, M_1, M_2, M_3, \|A_{11i}\|, \|A_{12i}\|, \|A_{21i}\|, \|A_{22i}\|, \|B_{1i}\|, \|B_{2i}\|, i = 0, \dots, l\}. \tag{3.1}$$

Multiplying the first equation of stationary system (1.1) by $e^{-\beta t}$, where β is an arbitrary positive number, and differentiating it, we obtain

$$\frac{d}{dt}(e^{-\beta t}x(t)) = -\beta e^{-\beta t}x(t) + \sum_{i=0}^{l} A_{11i}e^{-\beta t}x(t-ih) + \sum_{i=0}^{l} A_{12i}e^{-\beta t}y(t-ih) + \sum_{i=0}^{l} B_{1i}e^{-\beta t}u(t-ih), \quad t > 0. \tag{3.2}$$

Solving the second equation of system (1.1) "stepwise", we obtain the representation

$$\begin{aligned}
y(t) &= \sum_{j_0=0}^{l} A_{21j_0}x(t-j_0h) \\
&+ \sum_{i=1}^{T_t}\sum_{j_0=0}^{l}\sum_{j_1=1}^{l}\dots\sum_{\substack{j_i=1 \\ \left(\sum_{\mu=1}^{i} j_\mu h \le t\right)}}^{l} A_{22j_i}\dots A_{22j_1}A_{21j_0}x\left(t-\sum_{\mu=0}^{i} j_\mu h\right) \\
&+ \sum_{i=0}^{T_t}\sum_{j_0=1}^{l}\dots\sum_{\substack{j_i=1 \\ \left(\sum_{\mu=0}^{i} j_\mu h > t\right)}}^{l} A_{22j_i}\dots A_{22j_0}\psi\left(t-\sum_{\mu=0}^{i} j_\mu h\right) \\
&+ \sum_{j_0=0}^{l} B_{2j_0}u(t-j_0h) \\
&+ \sum_{i=1}^{T_t}\sum_{j_0=0}^{l}\sum_{j_1=1}^{l}\dots\sum_{\substack{j_i=1 \\ \left(\sum_{\mu=1}^{i} j_\mu h \le t\right)}}^{l} A_{22j_i}\dots A_{22j_1}B_{2j_0}u\left(t-\sum_{\mu=0}^{i} j_\mu h\right), \\
& t \ge 0.
\end{aligned}$$

Substituting this equality into (3.2) and integrating with respect to τ from 0 to t, we obtain the relation

$$\begin{aligned}
e^{-\beta t}x(t) &= x_0 - \beta\int_0^t e^{-\beta\tau}x(\tau)d\tau \\
&+ \int_0^t \sum_{k=0}^{l} A_{11k}e^{-\beta\tau}x(\tau-kh)d\tau \\
&+ \int_0^t \sum_{\substack{k=0 \\ (kh\le\tau)}}^{l} A_{12k}\sum_{j_0=0}^{l} A_{21j_0}e^{-\beta\tau}x(\tau-j_0h-kh)d\tau \\
&+ \int_0^t \sum_{k=0}^{l} A_{12k}\sum_{i=1}^{T_{\tau-kh}}\sum_{j_0=0}^{l}\sum_{j_1=1}^{l}\dots\sum_{\substack{j_i=1 \\ \left(kh+\sum_{\mu=1}^{i} j_\mu h \le \tau\right)}}^{l} A_{22j_i}\dots A_{22j_1}A_{21j_0} \times e^{-\beta\tau}x\left(\tau-\sum_{\mu=0}^{i} j_\mu h - kh\right)d\tau \\
&+ \int_0^t \sum_{k=0}^{l} A_{12k}\sum_{i=1}^{T_{\tau-kh}}\sum_{j_0=1}^{l}\dots\sum_{\substack{j_i=1 \\ \left(kh+\sum_{\mu=0}^{i} j_\mu h > \tau\right)}}^{l} A_{22j_i}\dots A_{22j_0} \times e^{-\beta\tau}\psi\left(\tau-\sum_{\mu=0}^{i} j_\mu h - kh\right)d\tau \\
&+ \int_0^t \sum_{\substack{k=0 \\ (kh\le\tau)}}^{l} A_{12k}\sum_{j_0=0}^{l} B_{2j_0}e^{-\beta\tau}u(\tau-j_0h-kh)d\tau \\
&+ \int_0^t \sum_{k=0}^{l} A_{12k}\sum_{i=1}^{T_{\tau-kh}}\sum_{j_0=0}^{l}\sum_{j_1=1}^{l}\dots\sum_{\substack{j_i=1 \\ \left(kh+\sum_{\mu=1}^{i} j_\mu h \le \tau\right)}}^{l} A_{22j_i}\dots A_{22j_1}B_{2j_0} \times e^{-\beta\tau}u\left(\tau-\sum_{\mu=0}^{i} j_\mu h - kh\right)d\tau \\
&+ \int_0^t \sum_{\substack{k=0 \\ (kh>\tau)}}^{l} A_{12k}e^{-\beta\tau}\psi(\tau-kh)d\tau \\
&+ \int_0^t \sum_{k=0}^{l} B_{1k}e^{-\beta\tau}u(\tau-kh)d\tau, \quad t > 0.
\end{aligned} \tag{3.3}$$

Evaluating (3.3) in the norm taking into account (3.1), we have

$$\|e^{-\beta t}x(t)\| \le \|x_0\| + \beta\int_0^t \|e^{-\beta\tau}x(\tau)\|d\tau$$

$$+ D\int_0^t \sum_{j_0=0}^{l} \|e^{-\beta\tau}x(\tau - j_0h)\|d\tau$$

$$+ D^2\int_0^t \sum_{k=0}^{\min\{l, T_\tau\}} \sum_{j_0=0}^{l} \|e^{-\beta\tau}x(\tau - j_0h - kh)\|d\tau$$

$$+ D^2\int_0^t \sum_{k=0}^{l} \sum_{i=1}^{T_{\tau-kh}} \underset{\left(kh + \sum_{\mu=1}^{i} j_\mu h \le \tau\right)}{\sum_{j_0=0}^{l} \sum_{j_1=1}^{l} \dots \sum_{j_i=1}^{l}} D^i$$

$$\times \left\| e^{-\beta\tau}x\left(\tau - \sum_{\mu=0}^{i} j_\mu h - kh\right)\right\| d\tau$$

$$+ D^2\int_0^t \sum_{k=0}^{l} \sum_{i=0}^{T_{\tau-kh}} \underset{\left(kh + \sum_{\mu=0}^{i} j_\mu h > \tau\right)}{\sum_{j_0=1}^{l} \dots \sum_{j_i=1}^{l}} D^i e^{-\beta\tau}$$

$$\times \left\| \psi\left(\tau - \sum_{\mu=0}^{i} j_\mu h - kh\right)\right\| d\tau$$

$$+ D^2\int_0^t \sum_{k=0}^{\min\{l, T_\tau\}} \sum_{j_0=0}^{l} \|e^{-\beta\tau}u(\tau - j_0h - kh)\|d\tau$$

$$+ D^2\int_0^t \sum_{k=0}^{l} \sum_{i=1}^{T_{\tau-kh}} \underset{\left(kh + \sum_{\mu=1}^{i} j_\mu h \le \tau\right)}{\sum_{j_0=0}^{l} \sum_{j_1=1}^{l} \dots \sum_{j_i=1}^{l}} D^i$$

$$\times \left\| e^{-\beta\tau}u\left(\tau - \sum_{\mu=0}^{i} j_\mu h - kh\right)\right\| d\tau$$

$$+ D\int_0^t \sum_{\substack{k=0 \\ (kh > \tau)}}^{l} \|e^{-\beta\tau}\psi(\tau - kh)\|d\tau$$

$$+ D\int_0^t \sum_{k=0}^{l} \|e^{-\beta\tau}u(\tau - kh)\|d\tau, \quad t > 0.$$

Extending the intervals of summation, we obtain

$$\|e^{-\beta t}x(t)\| \le \|x_0\| + \beta\int_0^t \|e^{-\beta\tau}x(\tau)\|d\tau$$

$$+ D\sum_{k=0}^{l}\int_0^t \|e^{-\beta\tau}x(\tau - kh)\|d\tau$$

$$+ D^2\sum_{k=0}^{l}\sum_{j_0=0}^{l}\int_0^t \|e^{-\beta\tau}x(\tau - j_0h - kh)\|d\tau$$

$$+ D^2\sum_{i=1}^{T_t}\sum_{j_0=0}^{l}\sum_{j_1=1}^{l}\dots\sum_{j_i=1}^{l} D^i$$

$$\times \sum_{k=0}^{l}\int_0^t \left\| e^{-\beta\tau}x\left(\tau - \sum_{\mu=0}^{i} j_\mu h - kh\right)\right\| d\tau$$

$$+ D^2\sum_{i=0}^{T_t}\sum_{j_0=1}^{l}\dots\sum_{j_i=1}^{l} D^i$$

$$\times \sum_{k=0}^{l} \int_0^{\min\left\{t,\, \sum_{\mu=0}^{i} j_\mu h + kh\right\}} e^{-\beta\tau}\left\| \psi\left(\tau - \sum_{\mu=0}^{i} j_\mu h - kh\right)\right\| d\tau$$

$$+ D^2\sum_{k=0}^{l}\sum_{j_0=0}^{l}\int_0^t e^{-\beta\tau}\|u(\tau - j_0h - kh)\|d\tau$$

$$+ D^2\sum_{i=1}^{T_t}\sum_{j_0=0}^{l}\sum_{j_1=1}^{l}\dots\sum_{j_i=1}^{l} D^i$$

$$\times \sum_{k=0}^{l}\int_0^t e^{-\beta\tau}\left\| u\left(\tau - \sum_{\mu=0}^{i} j_\mu h - kh\right)\right\| d\tau$$

$$+ D\sum_{k=0}^{l}\int_0^{\min\{t, kh\}} e^{-\beta\tau}\|\psi(\tau - kh)\|d\tau$$

$$+ D\sum_{k=0}^{l}\int_0^t e^{-\beta\tau}\|u(\tau - kh)\|d\tau, \quad t > 0.$$

Changing the variable of integration, we have

$$\|e^{-\beta t}x(t)\| \le \|x_0\| + \beta\int_0^t \|e^{-\beta s}x(s)\|ds$$

$$+ D\sum_{k=0}^{l}\int_{-kh}^{t-kh}\|e^{-\beta s}x(s)\|e^{-\beta kh}ds$$

$$+ D^2\sum_{k=0}^{l}\sum_{j_0=0}^{l}\int_{-j_0h-kh}^{t-j_0h-kh}\|e^{-\beta s}x(s)\|e^{-k\beta h}e^{-\beta j_0 h}ds$$

$$+ D^2\sum_{i=1}^{T_t}\sum_{j_0=0}^{l}\sum_{j_1=1}^{l}\dots\sum_{j_i=1}^{l}D^i$$

$$\times\sum_{k=0}^{l}\int_{-\sum_{\mu=0}^{i}j_\mu h-kh}^{t-\sum_{\mu=0}^{i}j_\mu h-kh}\|e^{-\beta s}x(s)\|e^{-k\beta h}e^{-\beta\sum_{\mu=0}^{i}j_\mu h}ds$$

$$+ D^2\sum_{i=0}^{T_t}\sum_{j_0=1}^{l}\dots\sum_{j_i=1}^{l}D^i$$

$$\times\sum_{k=0}^{l}\int_{-\sum_{\mu=0}^{i}j_\mu h-kh}^{\min\left\{t-\sum_{\mu=0}^{i}j_\mu h-kh,\,0\right\}}e^{-\beta s}\|\psi(s)\|e^{-\beta\sum_{\mu=0}^{i}j_\mu h}e^{-\beta kh}ds$$

$$+ D^2\sum_{k=0}^{l}\sum_{j_0=0}^{l}\int_{-j_0h-kh}^{t-j_0h-kh}e^{-\beta s}\|u(s)\|e^{-k\beta h}e^{-\beta j_0 h}ds$$

$$+ D^2\sum_{i=1}^{T_t}\sum_{j_0=0}^{l}\sum_{j_1=1}^{l}\dots\sum_{j_i=1}^{l}D^i$$

$$\times\sum_{k=0}^{l}\int_{-\sum_{\mu=0}^{i}j_\mu h-kh}^{t-\sum_{\mu=0}^{i}j_\mu h-kh}e^{-\beta s}\|u(s)\|e^{-k\beta h}e^{-\beta\sum_{\mu=0}^{i}j_\mu h}ds$$

$$+ D\sum_{k=0}^{l}\int_{-kh}^{\min\{t-kh,\,0\}}e^{-\beta s}\|\psi(s)\|e^{-kh}ds$$

$$+ D\sum_{k=0}^{l}\int_{-kh}^{t-kh}e^{-\beta s}\|u(s)\|e^{-\beta kh}ds, \quad t>0.$$

Extending the intervals of summation and integration, we obtain

$$\|e^{-\beta t}x(t)\| \le \|x_0\| + \beta\int_0^t\|e^{-\beta s}x(s)\|ds$$

$$+ D(l+1)\int_{-lh}^{0}\|\varphi(s)\|e^{-\beta s}ds + D(l+1)\int_0^t\|e^{-\beta s}x(s)\|ds$$

$$+ D^2(l+1)^2\int_0^t\|e^{-\beta s}x(s)\|ds$$

$$+ D^2(l+1)^2\int_{-lh}^{0}\|\varphi(s)\|e^{-\beta s}ds$$

$$+ D^2(l+1)^2\sum_{i=1}^{+\infty}(lDe^{-\beta h})^i\int_{-lh}^{0}\|\varphi(s)\|e^{-\beta s}ds$$

$$+ D^2(l+1)^2\sum_{i=1}^{+\infty}(lDe^{-\beta h})^i\int_0^t\|e^{-\beta s}x(s)\|ds$$

$$+ D^2(l+1)l\sum_{i=0}^{+\infty}(lDe^{-\beta h})^i\int_{-lh}^{0}\|\psi(s)\|e^{-\beta s}ds$$

$$+ D^2(l+1)^2\int_0^t\|u(s)\|e^{-\beta s}ds$$

$$+ D^2(l+1)^2\int_{-lh}^{0}e^{-\beta s}\|\xi(s)\|ds$$

$$+ D^2(l+1)^2\sum_{i=1}^{+\infty}(lDe^{-\beta h})^i\int_{-lh}^{0}\|\xi(s)\|e^{-\beta s}ds$$

$$+ D^2(l+1)^2\sum_{i=1}^{+\infty}(lDe^{-\beta h})^i\int_0^t\|u(s)\|e^{-\beta s}ds$$

$$+ D(l+1)\int_{-lh}^{0}e^{-\beta s}\|\psi(s)\|ds + D(l+1)\int_{-lh}^{0}\|\xi(s)\|e^{-\beta s}ds$$

$$+ D(l+1)\int_0^t\|u(s)\|e^{-\beta s}ds, \quad t>0.$$

Taking into account the estimates for initial data and (3.1), we have

$$\|e^{-\beta t}x(t)\| \le \|x_0\| + \Bigg(D^2(l+1) + D^3(l+1)^2$$

$$+ D^3(l+1)^2\sum_{i=1}^{+\infty}(lDe^{-\beta h})^i + D^3(l+1)l\sum_{i=0}^{+\infty}(lDe^{-\beta h})^i$$

$$+ D^3(l+1)^2 + D^3(l+1)^2\sum_{i=1}^{+\infty}(lDe^{-\beta h})^i + D^2(l+1)$$

$$+ D^2(l+1)\Bigg)\int_{-lh}^{0} e^{-\beta s}ds + \Bigg(D^3(l+1)^2$$

$$+ D^3(l+1)^2\sum_{i=1}^{+\infty}(lDe^{-\beta h})^i + D^2(l+1)\Bigg)\int_0^t e^{-(\beta-\sigma)s}ds$$

$$+ \Bigg(\beta + D(l+1) + D^2(l+1)^2$$

$$+ D^2(l+1)^2\sum_{i=1}^{+\infty}(lDe^{-\beta h})^i\Bigg)\int_0^t\|e^{-\beta s}x(s)\|ds \le \|x_0\|$$

$$+ \Bigg(3D^2(l+1) + 3D^3(l+1)^2\sum_{i=0}^{+\infty}(lDe^{-\beta h})^i\Bigg)\int_{-lh}^{0} e^{-\beta s}ds$$

$$+ \Bigg(D^2(l+1) + D^3(l+1)^2\sum_{i=0}^{+\infty}(lDe^{-\beta h})^i\Bigg)\int_0^t e^{-(\beta-\sigma)s}ds$$

$$+ \Bigg(\beta + D(l+1) + D^2(l+1)^2\sum_{i=0}^{+\infty}(lDe^{-\beta h})^i\Bigg)$$

$$\times\int_0^t\|e^{-\beta s}x(s)\|ds.$$

Take $\beta > 0$ such that $lDe^{-\beta h} < 1$ and $\beta - \sigma > 0$. Then,

$$\|e^{-\beta t}x(t)\| \le \|x_0\|$$

$$+ \left(3D^2(l+1) + 3D^3(l+1)^2\frac{1}{1-lDe^{-\beta h}}\right)\frac{1}{\beta}(e^{\beta lh} - 1)$$

$$+ \left(D^2(l+1) + D^3(l+1)^2\frac{1}{1-lDe^{-\beta h}}\right)$$

$$\times\frac{1}{\beta-\sigma}(1 - e^{-(\beta-\sigma)t})$$

$$+ \left(\beta + D(l+1) + D^2(l+1)^2\frac{1}{1-lDe^{-\beta h}}\right)\int_0^t\|e^{-\beta s}x(s)\|ds.$$

This yields

$$\|e^{-\beta t}x(t)\| \le K_\beta + N_\beta\int_0^t\|e^{-\beta s}x(s)\|ds,$$

where

$$K_\beta = \|x_0\|$$

$$+ \left(3D^2(l+1) + 3D^3(l+1)^2\frac{1}{1-lDe^{-\beta h}}\right)\frac{e^{\beta lh}}{\beta}$$

$$+ \left(D^2(l+1) + D^3(l+1)^2\frac{1}{1-lDe^{-\beta h}}\right)\frac{1}{\beta-\sigma},$$

$$N_\beta = \beta + D(l+1) + D^2(l+1)^2\frac{1}{1-lDe^{-\beta h}}.$$

By virtue of Lemma 2, we conclude that

$$\|e^{-\beta t}x(t)\| \le K_\beta e^{N_\beta t}$$

or $\|x(t)\| \le K_\beta e^{\alpha_1 t}$, $t > 0$, where $\alpha_1 = N_\beta + \beta$. Take a number L_1 such that $L_1 \ge K_\beta e^{\alpha_1 lh}$. Then, the inequality $\|x(t)\| \le L_1 e^{\alpha_1 t}$ is satisfied for all $t \in \mathbb{R}$, and the assertion of the theorem is proved for all $x(\cdot)$.

Now, we choose numbers L and α such that

$$\alpha > \alpha_1,$$
$$L \ge \max\left\{DL_1 + 3D^2l + D^2, \frac{D(L_1+D)(1+le^{-\alpha h})}{1-Dle^{-\alpha h}}\right\}. \tag{3.4}$$

In particular, the above relation yields that

$$L \ge DL_1 + D^2 + (DL_1 + DL + D^2)le^{-\alpha h}. \tag{3.5}$$

Relation (3.4) implies that $\|x(t)\| \le L_1 e^{\alpha_1 t} \le Le^{\alpha t}$ for all $t \in \mathbb{R}$.

Let us prove the exponential dependence of the growth rate for $y(t)$, $t \ge 0$, using the method of mathematical induction on $k : t \in [(k-1)h, kh)$, $k \in \mathbb{N}$. First of all, note that

$$\|y(t)\| = \|\psi(t)\| \le D \le L_1 e^{\alpha_1 t} \le Le^{\alpha t}, \quad t < 0. \tag{3.6}$$

At the first step, we have $k = 1, t \in [0, h)$, and the following estimate holds for $y(t)$:

$$\|y(t)\| = \left\| \sum_{i=0}^{l} A_{21i} x(t-ih) + \sum_{i=1}^{l} A_{22i} y(t-ih) + \sum_{i=0}^{l} B_{2i} u(t-ih) \right\|$$

$$\le D \sum_{i=0}^{l} \|x(t-ih)\| + D \sum_{i=1}^{l} \|\psi(t-ih)\|$$

$$+ D \sum_{i=0}^{l} \|u(t-ih)\| \le D\|x(t)\| + D \sum_{i=1}^{l} \|\varphi(t-ih)\|$$

$$+ D \sum_{i=1}^{l} \|\psi(t-ih)\| + D\|u(t)\| + D \sum_{i=1}^{l} \|\xi(t-ih)\|$$

$$\le DL_1 e^{\alpha_1 t} + D^2 l + D^2 l + D^2 e^{\sigma t} + D^2 l$$

$$\le (DL_1 + 3D^2 l + D^2) e^{\alpha t} \le L e^{\alpha t}.$$

The inequality $\|y(t)\| \le Le^{\alpha t}$ is satisfied for $k = 1$; therefore, the exponential estimate for the growth rate of $y(t)$ is valid for all $t < h$.

Assume now that the inequality $\|y(t)\| \le Le^{\alpha t}$, $t \in [(k-1)h, kh)$, is valid for any natural $k \le \nu$, $\nu \in \mathbb{N}$. We show that in this case it is also valid for $k = \nu + 1$. Indeed,

$$\|y(t)\| = \left\| \sum_{i=0}^{l} A_{21i} x(t-ih) + \sum_{i=1}^{l} A_{22i} y(t-ih) + \sum_{i=0}^{l} B_{2i} u(t-ih) \right\|$$

$$\le D \sum_{i=0}^{l} \|x(t-ih)\| + D \sum_{i=1}^{l} \|y(t-ih)\|$$

$$+ D \sum_{i=0}^{l} \|u(t-ih)\| \le DL_1 \sum_{i=0}^{l} e^{\alpha_1(t-ih)} + DL \sum_{i=1}^{l} e^{\alpha(t-ih)}$$

$$+ D^2 \sum_{i=0}^{l} e^{\sigma(t-ih)} = DL_1 e^{\alpha_1 t} + DL_1 e^{\alpha_1 t} \sum_{i=1}^{l} e^{-\alpha_1 ih}$$

$$+ DLe^{\alpha t} \sum_{i=1}^{l} e^{-\alpha ih} + D^2 e^{\sigma t} + D^2 e^{\sigma t} \sum_{i=1}^{l} e^{-\sigma ih}$$

$$\le (DL_1 + D^2) e^{\alpha t} + (DL_1 + DL + D^2) e^{\alpha t} \sum_{i=1}^{l} e^{-\alpha ih}$$

$$\le (DL_1 + D^2 + (DL_1 + DL + D^2) l e^{-\alpha h}) e^{\alpha t},$$

which, with due regard for (3.5) and the induction hypothesis, proves the estimate $\|y(t)\| \le Le^{\alpha t}$ for $t \ge 0$ and, taking into account inequality (3.6), for any $t \in \mathbb{R}$.

Theorem 1 is proved.

4. EXPANSION OF SOLUTIONS TO HDD SYSTEMS INTO SERIES IN POWERS OF SOLUTIONS TO THEIR DETERMINING EQUATIONS

Theorem 2. A solution to system (1.1) with initial conditions (1.2) for $t \ge 0$ exists, is unique, and can be represented at the points of continuity of $x(t)$, $y(t)$, and $u(t)$, $t \ge 0$, in the form of a series in powers of solutions to determining Eq. (2.1)

$$y(t) = \sum_{k=0}^{+\infty} \sum_{\substack{i \\ (t-ih \ge 0)}} X_{k+1}(ih) \int_0^{t-ih} \frac{(t-\tau-ih)^k}{k!} u(\tau) d\tau + x(t, x_0, \psi, \varphi, \xi, 0), \tag{4.1}$$

$$y(t) = \sum_{k=0}^{+\infty} \sum_{\substack{i \\ (t-ih \ge 0)}} Y_{k+1}(ih) \int_0^{t-ih} \frac{(t-\tau-ih)^k}{k!} u(\tau) d\tau + \sum_{\substack{i \\ (t-ih \ge 0)}} Y_0(ih) u(t-ih) + y(t, x_0, \psi, \varphi, \xi, 0). \tag{4.2}$$

Proof. Applying the Laplace transforms to system (1.2), (1.2), we obtain the system of linear algebraic equations

$$\left(pI_n - \sum_{i=0}^{l} A_{11i} e^{-pih}\right) \tilde{x}(p) = \sum_{i=0}^{l} A_{12i} e^{-pih} \tilde{y}(p) + \sum_{i=0}^{l} B_{1i} e^{-pih} \tilde{u}(p) + \left(x_0 + \sum_{i=0}^{l} \int_{-ih}^{0} e^{-p\tau} e^{-pih} (A_{11i}\varphi(\tau) + A_{12i}\psi(\tau) + B_{1i}\xi(\tau)) d\tau \right), \tag{4.3}$$

$$\left(I_m - \sum_{i=0}^{l} A_{22i} e^{-pih}\right) \tilde{y}(p) = \sum_{i=0}^{l} A_{21i} e^{-pih} \tilde{x}(p) + \sum_{i=0}^{l} B_{2i} e^{-pih} \tilde{u}(p) \tag{4.4}$$

$$+ \sum_{i=0}^{l} \int_{-ih}^{0} e^{-p\tau} e^{-pih} (A_{21i}\varphi(\tau) + A_{22i}\psi(\tau) + B_{2i}\xi(\tau)) d\tau,$$

where $\tilde{x}(\cdot) = l(x)(\cdot)$, $\tilde{y}(\cdot) = L(y)(\cdot)$, $\tilde{u}(\cdot) = L(u)(\cdot)$, and $\tilde{f}(\cdot)$ denotes the Laplace transform of $f(\tau)$, $\tau \in (0, +\infty)$ which is defined as follows:

$$\tilde{f}(p) = L(f)(p) = \int_0^{+\infty} f(\tau)e^{-p\tau}d\tau, \quad p \in \mathbb{C},$$

$\mathrm{Re}\,p > \alpha$, α is a positive number.

Expressing $\tilde{y}(p)$ from (4.4), we have

$$\tilde{y}(p) = \left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \times \left(\sum_{i=0}^{l} A_{21i}e^{-pih}\tilde{x}(p) + \sum_{i=0}^{l} B_{2i}e^{-pih}\tilde{u}(p)\right) + \left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \tag{4.5}$$
$$\times \sum_{i=0}^{l} \int_{-ih}^{0} e^{-p\tau}e^{-pih}(A_{21i}\varphi(\tau) + A_{22i}\psi(\tau) + B_{2i}\xi(\tau))d\tau.$$

Substituting (4.5) into (4.3) and solving with respect to $\tilde{x}(p)$, we obtain

$$\tilde{x}(p) = \left(pI_n - \sum_{i=0}^{l} A_{11i}e^{-pih} - \sum_{i=0}^{l} A_{12i}e^{-pih} \times \left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \sum_{i=0}^{l} A_{21i}e^{-pih}\right)^{-1} \left(\sum_{i=0}^{l} B_{1i}e^{-pih} + \sum_{i=0}^{l} A_{12i}e^{-pih}\left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \sum_{i=0}^{l} B_{2i}e^{-pih}\right)\tilde{u}(p)$$
$$+ \left(pI_n - \sum_{i=0}^{l} A_{11i}e^{-pih} - \sum_{i=0}^{l} A_{12i}e^{-pih} \times \left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \sum_{i=0}^{l} A_{21i}e^{-pih}\right)^{-1} \left(x_0 + \sum_{i=0}^{l} \int_{-ih}^{0} e^{-p\tau}e^{-pih}(A_{21i}\varphi(\tau) + A_{22i}\psi(\tau) + B_{2i}\xi(\tau))d\tau\right.$$
$$+ \sum_{i=0}^{l} A_{12i}e^{-pih}\left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \left.\times \sum_{i=0}^{l} \int_{-ih}^{0} e^{-p\tau}e^{-pih}(A_{21i}\varphi(\tau) + A_{22i}\psi(\tau) + B_{2i}\xi(\tau))d\tau\right).$$

Applying the formula for the sum of the matrix geometric series to the first term of this representation, we obtain

$$\tilde{x}(p) = \sum_{k=0}^{+\infty} \frac{1}{p^{k+1}}\left(\sum_{i=0}^{l} A_{11i}e^{-pih} + \sum_{i=0}^{l} A_{12i}e^{-pih} \times \left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \sum_{i=0}^{l} A_{21i}e^{-pih}\right)^{k} \left(\sum_{i=0}^{l} B_{1i}e^{-pih} + \sum_{i=0}^{l} A_{12i}e^{-pih}\left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \sum_{i=0}^{l} B_{2i}e^{-pih}\right)\tilde{u}(p)$$
$$+ \sum_{k=0}^{+\infty} \frac{1}{p^{k+1}}\left(\sum_{i=0}^{l} A_{11i}e^{-pih} + \sum_{i=0}^{l} A_{12i}e^{-pih} \times \left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \sum_{i=0}^{l} A_{21i}e^{-pih}\right)^{k} \left(x_0 \right. \tag{4.6}$$
$$+ \sum_{i=0}^{l} \int_{-ih}^{0} e^{-p\tau}e^{-pih}(A_{21i}\varphi(\tau) + A_{22i}\psi(\tau) + B_{2i}\xi(\tau))d\tau + \sum_{i=0}^{l} A_{12i}e^{-pih}\left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1}$$
$$\left.\times \sum_{i=0}^{l} \int_{-ih}^{0} e^{-p\tau}e^{-pih}(A_{21i}\varphi(\tau) + A_{22i}\psi(\tau) + B_{2i}\xi(\tau))d\tau\right).$$

Then, taking into account (4.6), we can refine formula (4.5) as

$$\tilde{y}(p) = \left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \sum_{i=0}^{l} A_{21i}e^{-pih} \times \sum_{k=0}^{+\infty} \frac{1}{p^{k+1}}\left(\sum_{i=0}^{l} A_{11i}e^{-pih} + \sum_{i=0}^{l} A_{12i}e^{-pih} \times \left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \sum_{i=0}^{l} A_{21i}e^{-pih}\right)^{k} \left(\sum_{i=0}^{l} B_{1i}e^{-pih} + \sum_{i=0}^{l} A_{12i}e^{-pih}\left(I_m - \sum_{i=0}^{l} A_{22i}e^{-pih}\right)^{-1} \sum_{i=0}^{l} B_{2i}e^{-pih}\right)\tilde{u}(p)$$

$$+\left(I_m-\sum_{i=0}^{l}A_{22i}e^{-pih}\right)^{-1}\sum_{i=0}^{l}B_{2i}e^{-pih}\tilde{u}(p)$$

$$+\left(I_m-\sum_{i=0}^{l}A_{22i}e^{-pih}\right)^{-1}\sum_{i=0}^{l}A_{21i}e^{-pih}$$

$$\times\sum_{k=0}^{+\infty}\frac{1}{p^{k+1}}\left(\sum_{i=0}^{l}A_{11i}e^{-pih}+\sum_{i=0}^{l}A_{12i}e^{-pih}\right. \tag{4.7}$$

$$\times\left.\left(I_m-\sum_{i=0}^{l}A_{22i}e^{-pih}\right)^{-1}\sum_{i=0}^{l}A_{21i}e^{-pih}\right)^k\left(x_0\right.$$

$$+\sum_{i=0}^{l}\int_{-ih}^{0}e^{-p\tau}e^{-pih}(A_{11i}\varphi(\tau)+A_{12i}\psi(\tau)+B_{1i}\xi(\tau))d\tau$$

$$+\sum_{i=0}^{l}A_{12i}e^{-pih}\left(I_m-\sum_{i=0}^{l}A_{22i}e^{-pih}\right)^{-1}$$

$$\left.\times\sum_{i=0}^{l}\int_{-ih}^{0}e^{-p\tau}e^{-pih}(A_{21i}\varphi(\tau)+A_{22i}\psi(\tau)+B_{2i}\xi(\tau))d\tau\right)$$

$$+\left(I_m-\sum_{i=0}^{l}A_{22i}e^{-pih}\right)^{-1}$$

$$\times\sum_{i=0}^{l}\int_{-ih}^{0}e^{-p\tau}e^{-pih}(A_{21i}\varphi(\tau)+A_{22i}\psi(\tau)+B_{2i}\xi(\tau))d\tau.$$

Applying identities (2.2)–(2.4) to (4.6), (4.7), we obtain the representations

$$\tilde{x}(p) = \sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}X_{k+1}(jh)\frac{e^{-pjh}}{p^{k+1}}\tilde{u}(p) + \int_0^{+\infty}x(t, x_0, \varphi, \psi, \xi, 0)e^{-pt}dt, \tag{4.8}$$

$$\tilde{y}(p) = \sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}Y_{k+1}(jh)\frac{e^{-pjh}}{p^{k+1}}\tilde{u}(p) + \sum_{j=0}^{+\infty}Y_0(jh)e^{-pjh}\tilde{u}(p)+\int_0^{+\infty}y(t, x_0, \varphi, \psi, \xi, 0)e^{-pt}dt. \tag{4.9}$$

As regards the initial relations, we make sure that Theorem 2 is valid in the case of continuous controls.

Remark 2. In formulas (4.1), (4.2), the components of the solution

$$\begin{bmatrix}x(t;\, x_0, \varphi, \psi, \xi, 0)\\ y(t;\, x_0, \varphi, \psi, \xi, 0)\end{bmatrix}$$

do not have a completed form such as

$$\begin{bmatrix}x(t;\, 0, 0, 0, 0, u)\\ y(t;\, 0, 0, 0, 0, u)\end{bmatrix}.$$

However, if $\psi(\tau) = 0$, $\varphi(\tau) = 0$, $\xi(\tau) = 0$, $\tau \in [-lh, 0]$, then it can be checked directly that the following expansions are valid:

$$x(t;\, x_0, 0, 0, 0, 0) = \sum_{k=0}^{+\infty}\sum_{\substack{i\\(t-ih\ge 0)}}X^0_{k+1}(ih)\frac{(t-ih)^k}{k!}x_0, \quad t\ge 0,$$

$$y(t;\, x_0, 0, 0, 0, 0) = \sum_{k=0}^{+\infty}\sum_{\substack{i\\(t-ih\ge 0)}}Y^0_{k+1}(ih)\frac{(t-ih)^k}{k!}x_0, \quad t\ge 0,$$

where $0^0 = 1$; $X^0_k(t)$, $Y^0_k(t)$ is the solution to determining Eq. (2.1) for $r = n$ with the matrices $B_{10} = I_n$, $B_{1i} = 0$, $B_{20} = 0$, $B_{2i} = 0$, $(i = 0, 1, \ldots, l)$.

5. INTEGRAL REPRESENTATIONS OF SOLUTIONS TO HDD SYSTEMS USING SOLUTIONS TO ADJOINT SYSTEMS

Assume that

$$-\frac{dX^*(t)}{dt}+\sum_{j=0}^{l}(X^*(t-jh)A_{11j}+Y^*(t-jh)A_{21j}) = 0, \quad t\ge 0,\quad t\ne kh; \tag{5.1}$$

$$Y^*(t) = \sum_{j=0}^{l}(X^*(t-jh)A_{12j}+Y^*(t-jh)A_{22j}), \quad t\ge 0; \tag{5.2}$$

$$X^*(kh)-X^*(kh-0) = \sum_{j=k-l}^{k}Z^*(jh)A_{21k-j}; \tag{5.3}$$

$$Z^*(kh) = \sum_{j=k-l}^{k-1}Z^*(jh)A_{22k-j} \tag{5.4}$$

for $t \ge 0$; $k = 1, \ldots, T_t$, and

$$Y^*(t) = 0,\quad X^*(t) = 0,\quad Z^*(t) = 0,\quad t<0. \tag{5.5}$$

In [5], integral representations of the type of the Cauchy formula were obtained for linear differential–algebraic systems with one delay. A generalization of this result to the case of stationary systems with commeasurable delays is the following theorem.

Theorem 3. A solution to system (1.1), (1.2) corresponding to the admissible control $u(\tau)$, $\tau \in [0, t]$, exists, is unique, and can be represented in the form

$$x(t) = \int_0^t \sum_{j=0}^{l} (X_x^*(t-\tau-jh)B_{1j} + Y_x^*(t-\tau-jh)B_{2j}) \times u(\tau)d\tau + x(t;\, x_0, \varphi, \psi, \xi, 0), \quad t \ge 0, \tag{5.6}$$

with the initial conditions $X_x^*(0) = X^*(0) = I_n$ and $Z_x^*(0) = Z^*(0) = 0 \in \mathbb{R}^{n \times m}$; and

$$y(t) = \int_0^t \sum_{j=0}^{l} (X_y^*(t-\tau-jh)B_{1j} + Y_y^*(t-\tau-jh)B_{2j}) \times u(\tau)d\tau + \sum_{j=0}^{T_t} \sum_{k=j-l}^{j} Z_y^*(kh)B_{2j-k}u(t-jh) + y(t;\, x_0, \varphi, \psi, \xi, 0), \quad t \ge 0, \tag{5.7}$$

with the initial conditions $X_y^*(0) = X^*(0) = A_{210} \in \mathbb{R}^{m \times n}$ and $Z_y^*(0) = Z^*(0) = I_m$.

Here, the n-vector function $x(t; x_0, \xi(\tau), \psi(\tau), \varphi(\tau), 0)$ and the m-vector function $y(t; x_0, \xi(\tau), \psi(\tau), \varphi(\tau), 0)$, $\tau \in [-lh, 0]$, depend only on the initial conditions and have the form

$$\begin{aligned} x(t;\, x_0, \varphi, \psi, \xi, 0) &= \sum_{j=0}^{l} \int_{-jh}^{0} (X_x^*(t-\tau-jh)A_{12j} + Y_x^*(t-\tau-jh)A_{22j})\psi(\tau)d\tau \\ &+ \sum_{j=0}^{l} \int_{-jh}^{0} (X_x^*(t-\tau-jh)B_{1j} + Y_x^*(t-\tau-jh)B_{2j})\xi(\tau)d\tau \\ &+ \sum_{j=0}^{l} \int_{-jh}^{0} (X_x^*(t-\tau-jh)A_{11j} + Y_x^*(t-\tau-jh)A_{21j})\varphi(\tau)d\tau + X_x^*(t)x_0, \end{aligned} \tag{5.8}$$

$$\begin{aligned} y(t;\, x_0, \varphi, \psi, \xi, 0) &= \sum_{j=0}^{l} \int_{-jh}^{0} (X_y^*(t-\tau-jh)A_{12j} + Y_y^*(t-\tau-jh)A_{22j})\psi(\tau)d\tau \\ &+ \sum_{j=0}^{l} \int_{-jh}^{0} (X_y^*(t-\tau-jh)B_{1j} + Y_y^*(t-\tau-jh)B_{2j})\xi(\tau)d\tau \\ &+ \sum_{j=0}^{l} \int_{-jh}^{0} (X_y^*(t-\tau-jh)A_{11j} + Y_y^*(t-\tau-jh)A_{21j})\varphi(\tau)d\tau + X_y^*(t)x_0 \\ &+ \sum_{j=T_t+1}^{T_t+l} \sum_{k=j-l}^{T_t} Z_y^*(kh)(A_{21j-k}\varphi(t-jh) + A_{22j-k}\psi(t-jh) + B_{2j-k}\xi(t-jh)), \end{aligned} \tag{5.9}$$

respectively, where the matrix functions $X^*(\cdot)$, $Z^*(\cdot)$, and $Y^*(\cdot)$ are solutions to adjoint system (5.1)–(5.5).

6. CORRELATION OF SOLUTIONS TO ADJOINT SYSTEMSAND SOLUTIONS TO DETERMINING EQUATIONS

Lemma 3. $Z^*(0) = I_m$ and $X^*(0) = A_{210}$, the following identities are satisfied:

$$\sum_{k=i-l}^{i} Z^*(kh)B_{2i-k} = Y_0(ih), \quad i = 0, 1, \ldots; \tag{6.1}$$

$$\sum_{k=0}^{+\infty} (X^*(kh) - X^*(kh-0))\omega^k \equiv \left(I_m - \sum_{i=1}^{l} A_{22i}\omega^i\right)^{-1} \sum_{i=1}^{l} A_{21i}\omega^i \tag{6.2}$$

for $|\omega| \le \omega_1$, where ω_1 is a sufficiently small positive number.

The proof of Lemma 3 is performed by the method of mathematical induction using solutions to the determining equation for $k = 0$ and formulas (5.3), (5.4).

Lemma 4. At the points of continuity of $X^*(\cdot)$ and $Y^*(\cdot)$, the following identities are satisfied:

$$\sum_{i=0}^{l} (X^*(t-ih)B_{1i} + Y^*(t-ih)B_{2i}) = \sum_{k=0}^{+\infty} \sum_{\substack{i=0 \\ (t-ih \ge 0)}}^{+\infty} X_{k+1}(ih)\frac{(t-ih)^k}{k!} \tag{6.3}$$

for $Z^*(0) = 0$ and $X^*(0) = I_n$; and

$$\sum_{i=0}^{l}(X^*(t-ih)B_{1i} + Y^*(t-ih)B_{2i}) = \sum_{k=0}^{+\infty}\sum_{\substack{i=0\\(t-ih\geq 0)}}^{+\infty} Y_{k+1}(ih)\frac{(t-ih)^k}{k!} \tag{6.4}$$

for $Z^*(0) = I_m$ and $X^*(0) = A_{210}$.

Proof. Applying the Laplace transform to adjoint system (5.1)–(5.5) and using the notation $\tilde{X}(\cdot) = L(X^*)(\cdot)$, $\tilde{Y}(\cdot) = L(Y^*)(\cdot)$, we obtain

$$\int_0^{+\infty}\dot{X}^*(\tau)e^{-p\tau}d\tau = \sum_{i=0}^{l}\tilde{X}(p)A_{11i}e^{-iph} + \sum_{i=0}^{l}\tilde{Y}(p)A_{21i}e^{-iph}, \tag{6.5}$$

$$\tilde{Y}(p) = \sum_{i=0}^{l}\tilde{X}(p)A_{12i}e^{-pih}\left(I_m - \sum_{i=1}^{l}A_{22i}e^{-iph}\right)^{-1}. \tag{6.6}$$

Integrating the left-hand side of (6.5) by parts, we find that

$$\int_0^{+\infty}\dot{X}^*(\tau)e^{-p\tau}d\tau = \sum_{k=0}^{+\infty}\left(\int_{kh}^{(k+1)h}\dot{X}^*(\tau)e^{-p\tau}d\tau\right)$$

$$= -X^*(0) + \sum_{k=1}^{+\infty}(X^*(kh-0) - X^*(kh))e^{-kph}$$

$$+ p\sum_{k=0}^{+\infty}\int_{kh}^{(k+1)h}X^*(\tau)e^{-p\tau}d\tau$$

$$= -X^*(0) + \sum_{k=1}^{+\infty}(X^*(kh-0) - X^*(kh))e^{-kph}$$

$$+ p\int_0^{+\infty}X^*(\tau)e^{-p\tau}d\tau = -X^*(0)$$

$$+ \sum_{k=1}^{+\infty}(X^*(kh-0) - X^*(kh))e^{-kph} + p\tilde{X}(p).$$

Substituting the obtained representation and (6.6) into (6.5), we have

$$\sum_{k=0}^{+\infty}(X^*(kh) - X^*(kh-0))e^{-pkh}$$

$$= \tilde{X}(p)\left(pI_n - \sum_{i=0}^{l}A_{11i}e^{-pih}\right.$$

$$\left. - \sum_{i=0}^{l}A_{12i}e^{-pih}\left(I_m - \sum_{i=0}^{l}A_{22i}e^{-pih}\right)^{-1}\sum_{i=0}^{l}A_{21i}e^{-pih}\right)$$

or

$$\tilde{X}(p) = \sum_{k=0}^{+\infty}(X^*(kh) - X^*(kh-0))e^{-pkh}$$

$$\times\left(pI_n - \sum_{i=0}^{l}A_{11i}e^{-pih}\right.$$

$$\left. - \sum_{i=0}^{l}A_{12i}e^{-pih}\left(I_m - \sum_{i=0}^{l}A_{22i}e^{-pih}\right)^{-1}\sum_{i=0}^{l}A_{21i}e^{-pih}\right)^{-1} \tag{6.7}$$

$$= \sum_{k=0}^{+\infty}(X^*(kh) - X^*(kh-0))e^{-pkh}$$

$$\times\sum_{k=0}^{+\infty}\frac{1}{p^{k+1}}\left(\sum_{i=0}^{l}A_{11i}e^{-pih}\right.$$

$$\left. + \sum_{i=0}^{l}A_{12i}e^{-pih}\left(I_m - \sum_{i=0}^{l}A_{22i}e^{-pih}\right)^{-1}\sum_{i=0}^{l}A_{21i}e^{-pih}\right)^k.$$

Then, (6.6) can be refined as

$$\tilde{Y}(p) = \sum_{k=0}^{+\infty}(X^*(kh) - X^*(kh-0))e^{-pkh}$$

$$\times\sum_{k=0}^{+\infty}\frac{1}{p^{k+1}}\left(\sum_{i=0}^{l}A_{11i}e^{-pih}\right.$$

$$\left. + \sum_{i=0}^{l}A_{12i}e^{-pih}\left(I_m - \sum_{i=0}^{l}A_{22i}e^{-pih}\right)^{-1}\sum_{i=0}^{l}A_{21i}e^{-pih}\right)^k \tag{6.8}$$

$$\times\sum_{i=0}^{l}A_{12i}e^{-iph}\left(I_m - \sum_{i=1}^{l}A_{22i}e^{-iph}\right)^{-1}.$$

Multiplying both sides of Eqs. (6.7) and (6.8) from the right by

$$\sum_{i=0}^{l} B_{1i} e^{-pih} \text{ и } \sum_{i=0}^{l} B_{2i} e^{-pih},$$

respectively, and summing the results, we obtain the relation

$$\begin{aligned} &\tilde{X}(p)\sum_{i=0}^{l} B_{1i} e^{-pih} + \tilde{Y}(p)\sum_{i=0}^{l} B_{2i} e^{-pih} \\ &= \sum_{k=0}^{+\infty} (X^*(kh) - X^*(kh-0)) e^{-pkh} \\ &\times \sum_{k=0}^{+\infty} \frac{1}{p^{k+1}} \left(\sum_{i=0}^{l} A_{11i} e^{-pih} \right. \\ &+ \left. \sum_{i=0}^{l} A_{12i} e^{-pih} \left(I_m - \sum_{i=0}^{l} A_{22i} e^{-pih} \right)^{-1} \sum_{i=0}^{l} A_{21i} e^{-pih} \right)^k \\ &\times \left(\sum_{i=0}^{l} B_{1i} e^{-pih} \right. \\ &+ \left. \sum_{i=0}^{l} A_{12i} e^{-pih} \left(I_m - \sum_{i=0}^{l} A_{22i} e^{-pih} \right)^{-1} \sum_{i=0}^{l} B_{2i} e^{-pih} \right). \end{aligned} \tag{6.9}$$

Assigning $Z^*(0) = 0$ and $X^*(0) = I_n$, we conclude from Eqs. (5.3), (5.4) that

$$\sum_{k=0}^{+\infty} (X^*(kh) - X^*(kh-0)) e^{-pkh} = I_n.$$

Thus, taking into account identity (2.2), we see that (6.9) takes the form

$$\begin{aligned} &\tilde{X}(p)\sum_{i=0}^{l} B_{1i} e^{-pih} + \tilde{Y}(p)\sum_{i=0}^{l} B_{2i} e^{-pih} \\ &= \sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty} X_{k+1}(jh) \frac{e^{-pjh}}{p^{k+1}}. \end{aligned} \tag{6.10}$$

Applying the inverse Laplace transform to equality (6.10), we prove identity (6.3) of the lemma.

For $Z^*(0) = I_m$ and $X^*(0) = A_{210}$, taking into account (6.2) and (2.3), we represent equality (6.9) as follows:

$$\begin{aligned} &\tilde{X}(p)\sum_{i=0}^{l} B_{1i} e^{-pih} + \tilde{Y}(p)\sum_{i=0}^{l} B_{2i} e^{-pih} \\ &= \sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty} Y_{k+1}(jh) \frac{e^{-pjh}}{p^{k+1}}. \end{aligned} \tag{6.11}$$

Now, going back to the initial relation, we obtain identity (6.4) of the lemma. The proof of the lemma is completed.

Remark 3. The expansion of solutions to the adjoint system in series in powers of solutions to the determining equation of the hybrid system, which is proposed in Lemma 4, can be interpreted as a generalization of the known formula of expansion of the fundamental matrix of solutions for ordinary nondegenerate systems with a retarded argument into a series in powers of the solutions to the determining equation [16].

On the basis of the obtained result, we prove the assertion of Theorem 3 for $u(\cdot) \in PC([0, +\infty), \mathbb{R}^r)$. Multiplying both sides of equality (6.10) by $u(p)$ and applying the convolution theorem, we have

$$\begin{aligned} &\int_0^t \sum_{i=0}^{l} (X_x^*(t-\tau-ih)B_{1i} + Y_x^*(t-\tau-ih)B_{2i})u(\tau)d\tau \\ &= \sum_{k=0}^{+\infty} \sum_{\substack{i \\ (t-ih>0)}} X_{k+1}(ih) \int_0^{t-ih} \frac{(t-\tau-ih)^k}{k!} u(\tau)d\tau \end{aligned}$$

for $Z_x^*(0) = Z^*(0)$ and $X_x^*(0) = X^*(0) = I_n$.

From this, taking into account (5.6) and $Z_x^*(kh) = 0$, $k = 0, 1, \ldots$ (which follows from the initial conditions and Eq. (5.4)), we obtain relation (4.1).

Similarly, on the basis of equality (6.11), we can conclude that

$$\begin{aligned} &\int_0^t \sum_{i=0}^{l} (X_y^*(t-\tau-ih)B_{1i} + Y_y^*(t-\tau-ih)B_{2i})u(\tau)d\tau \\ &= \sum_{k=0}^{+\infty} \sum_{\substack{i \\ (t-ih>0)}} Y_{k+1}(ih) \int_0^{t-ih} \frac{(t-\tau-ih)^k}{k!} u(\tau)d\tau \end{aligned}$$

for $Z_y^*(0) = Z^*(0) = I_m$ and $X_y^*(0) = X^*(0) = A_{210}$.

This equality and identity (6.1) in representation (5.7) prove the validity of (4.2).

Theorem 3 in the case of piecewise-continuous control is proved.

Remark 4. In the general case, the validity of conditions (4.1), (4.2) can be verified directly by substituting them into the system. The proof given above is in our opinion interesting in itself from the point of view of investigation of the properties of the adjoint system.

Corollary 1. For any control $u(\cdot) \in PC([0, +\infty), \mathbb{R}^r)$, a solution to system (1.1) with zero initial conditions exists, is unique, and can be represented in the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \sum_{k=0}^{+\infty} \sum_{j=0}^{T_t} \int\limits_{\max\{0,\, t-(j+1)h\}}^{t-jh} \sum_{i=0}^{j} \begin{bmatrix} X_{k+1}(ih) \\ Y_{k+1}(ih) \end{bmatrix} \times \frac{(t-\tau-ih)^k}{k!} u(\tau)d\tau + \sum_{j=0}^{T_t} \begin{bmatrix} 0 \\ Y_0(jh) \end{bmatrix} u(t-jh), \quad t \ge 0. \tag{6.12}$$

7. EXAMPLE

Let us consider the control system

$$\begin{aligned} \dot{x}(t) &= x(t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} y(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} y(t-h) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \end{aligned} \tag{7.1}$$

with zero initial conditions.

From determining Eq. (2.1), we successively obtain

$$Y_0(0) = A_{22}Y_0(-h) + B_2U_0(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$Y_0(h) = A_{22}Y_0(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$Y_0(2h) = A_{22}Y_0(h) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$Y_0(3h) = A_{22}Y_0(2h) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$Y_0(jh) = A_{22}Y_0((j-1)h) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad j \ge 3,$$

$$Y_k(0) = A_{22}Y_k(-h) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \dots,$$

$$Y_k(jh) = A_{22}Y_k(jh-h) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad k \ge 1, \quad j \ge 0.$$

$$X_0(jh) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad j \ge 0,$$

$$X_1(0) = A_{12}Y_0(0) + B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$X_1(h) = A_{12}Y_0(h) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$X_1(2h) = A_{12}Y_0(2h) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$X_1(3h) = A_{12}Y_0(3h) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\dots X_1(jh) = A_{12}Y_0(jh) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad j \ge 3,$$

$$X_2(0) = A_{12}Y_1(0) + X_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$X_2(h) = A_{12}Y_1(h) + X_1(h) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$X_2(2h) = A_{12}Y_1(2h) + X_1(2h) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$X_2(3h) = A_{12}Y_1(3h) + X_1(3h) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$X_2(jh) = A_{12}Y_1(jh) + X_1(jh) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad j \ge 3,$$

$$X_k(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X_k(h) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad X_k(2h) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$X_k(jh) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad k = 1, 2, \dots, \quad j \ge 3.$$

According to representation (6.12), we have

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \sum_{k=0}^{+\infty} \int_0^t \frac{(t-\tau)^k}{k!} u(\tau) d\tau + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

for $0 \le t < h$.

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \int_{t-h}^{t} \sum_{k=0}^{+\infty} \frac{(t-\tau)^k}{k!} u(\tau) d\tau$$

$$+ \int_0^{t-h} \sum_{k=0}^{+\infty} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{(t-\tau)^k}{k!} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{(t-\tau-h)^k}{k!} \right) u(\tau) d\tau$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t-h) \quad \text{for} \quad h \le t < 2h;$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \int_{t-h}^{t} \sum_{k=0}^{+\infty} \frac{(t-\tau)^k}{k!} u(\tau) d\tau$$

$$+ \int_0^{t-h} \sum_{k=0}^{+\infty} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{(t-\tau)^k}{k!} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{(t-\tau-h)^k}{k!} \right) u(\tau) d\tau$$

$$+ \sum_{j=2}^{T_t} \int_{\max\{0,\, t-(j+1)h\}}^{t-jh} \sum_{k=0}^{+\infty} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{(t-\tau)^k}{k!} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{(t-\tau-h)^k}{k!} \right.$$

$$\left. + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{(t-\tau-2h)^k}{k!} \right) u(\tau) d\tau + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t-h) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t-2h) \quad \text{for} \quad t \ge 2.$$

CONCLUSIONS

Thus, in this work the exponential estimate for the growth rate of solutions to the stationary HDD systems is proved. This allows one to apply the Laplace transform to such systems. The algebraic properties of solutions to the determining equations are studied. This provides considerable simplification of the representation of solutions to the HDD systems in the frequency domain. An expansion of solutions to stationary adjoint systems into series in powers of solutions to the determining equations is given. For controls in the class of continuous and piecewise-continuous functions, representations of solutions to the stationary controlled HDD systems are obtained in the form of series in powers of solutions to their determining equations. This completes the first part of the work. The solutions found in it have important applications in qualitative control theory in HDD systems, in particular, in the investigation of problems of controllability in such systems. This will be discussed in the second part of this work.

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