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CONTROL THEORY

Solution Representations and Relative Controllability of Linear Differential Algebraic Systems with Several Delays

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The study of real-life physical processes encounters not only differential but also algebraic (functional) relations. Such processes are described by differential algebraic systems [1], which are known as hybrid [2–4]. It should be noted that the term "hybrid systems" is overloaded. Generally speaking, hybridity means that the nature of the process under study or the methods used for its description and analysis are inhomogeneous. At present, especially in English-language publications, the term "hybrid" refers primarily to discrete–continuous systems or systems with logical variables [5–7]. We consider differential algebraic (hybrid) delay systems to which, in particular, certain standard types of linear discrete–continuous systems can be reduced.

Consider the system

$$\dot{x}(t) = \sum_{i=0}^{l}(A_{11i}(t)x(t-ih) + A_{12i}(t)y(t-ih) + B_{1i}u(t-ih)), \tag{1}$$

$$y(t) = \sum_{i=0}^{l}(A_{21i}(t)x(t-ih) + A_{22i}(t)y(t-ih) + B_{2i}u(t-ih)), \quad t \ge t_0, \tag{2}$$

with the initial conditions

$$\begin{gathered} x(t_0+0) = x(t_0) = x_0 \in \mathbf{R}^n, \quad x(\tau) = \varphi(\tau), \\ y(\tau) = \psi(\tau), \quad u(\tau) = \xi(\tau), \quad \tau \in [t_0 - lh, t_0). \end{gathered} \tag{3}$$

Here, $x(t) \in \mathbf{R}^n$; $y(t) \in \mathbf{R}^m$; $u(t) \in \mathbf{R}^r$; $l \in N$; $A_{220}(t) = 0$; $0 < h$ is a constant delay; and the elements of the matrix functions $A_{11i}(t)$, $A_{12i}(t)$, $A_{21i}(t)$, $A_{22i}(t)$, $B_{1i}(t)$, and $B_{2i}(t)$ ($i = 0, 1, \ldots, l$), $t \ge t_0$ and of the vector functions $\psi(t)$, $\varphi(t)$, and $\xi(t)$ ($t \in [t_0 - lh, t_0]$) are piecewise continuous functions. The external action $u(t)$ for $t \ge t_0$ is a piecewise continuous r-vector function (admissible control). The right-hand derivative is considered at $t = t_0$ in (1).

The solution $x(t) = x(t; t_0, x_0, \varphi, \psi, \xi, u)$, $y(t) = y(t; t_0, x_0, \varphi, \psi, \xi, u)$ ($t \ge t_0$) to system (1), (2) with initial conditions (3) and an admissible control $u = u(t)$ ($t \ge t_0$) is defined as arbitrary vector functions $x(t)$ and $y(t)$ ($t \ge t_0$) that satisfy the second equation and satisfy the first equation in the system almost everywhere for $t \ge t_0$. Here, it is assumed that $x(\cdot)$ is continuous and $y(\cdot)$ is piecewise continuous on the interval $[0, +\infty)$.

For system (1)–(3), we obtain solution representations (variation-of-constants formulas), which were previously known for ordinary systems and systems with a delayed argument [8–10]. The fundamental difference of the new representations is that the adjoint system of the equation involves jump equations. For stationary systems (1)–(3), solutions' representations are given in the form of series over solutions to the systems' determining equations, which is a substantial generalization of similar formulas for ordinary systems (via a series expansion of a matrix exponential function). The results are applied to the investigation of the relative controllability of differential algebraic systems with delay. We establish several useful algebraic properties of solutions to the corresponding determining equations, which are used to find a finite number of generators in the linear span of columns of the controllability matrix.

Suppose that matrix-valued functions $X^*(t, \tau)$, $Z^*(t, \tau)$, and $Y^*(t, \tau)$ are the solutions to the reverse-time adjoint system

$$\frac{\partial X^*(t,\tau)}{\partial \tau} + \sum_{i=0}^{l}(X^*(t, \tau+ih)A_{11i}(\tau+ih) + Y^*(t, \tau+ih)A_{21i}(\tau+ih)) = 0 \tag{4}$$

almost everywhere for $\tau \le t$;

$$Y^*(t,\tau) = \sum_{i=0}^{l}(X^*(t, \tau+ih)A_{12i}(\tau+ih) + Y^*(t, \tau+ih)A_{22i}(\tau+ih)), \quad \tau \le t; \tag{5}$$

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$$X^*(t, t-kh-0) - X^*(t, t-kh+0) = \sum_{i=k-l}^{k} Z^*(t, t-ih)A_{21k-i}(t-th); \tag{6}$$

$$Z^*(t, t-kh) = \sum_{i=k-l}^{k} Z^*(t, t-ih)A_{22k-i}(t-ih), \quad k = 1, 2, \dots, \quad T_t = \left[\frac{t-t_0}{h}\right], \tag{7}$$

where

$$Y^*(t, \tau) = 0, \quad X^*(t, \tau) = 0, \quad Z^*(t, \tau) = 0, \quad \tau > t. \tag{8}$$

The symbol $[z]$ in (7) stands for the integer of part of z.

Theorem 1. *System* (1)–(3) *with an admissible control* $u(\tau)$ ($\tau \in [t_0, t]$) *has a unique solution* $x(t) = x(t; t_0, x_0, \varphi, \psi, \xi, u)$, $y(t) = y(t; t_0, x_0, \varphi, \psi, \xi, u)$ ($t \ge t_0$) *given by the formula*

$$\begin{aligned}
&\int_{t_0}^{t}\sum_{i=0}^{l}(X^*(t, \tau+ih)B_{1i}(\tau+ih) + Y^*(t, \tau+ih)B_{2i}(\tau+ih))u(\tau)d\tau \\
&+ \sum_{k=0}^{T_t}\sum_{i=k-l}^{k} Z^*(t, t-ih)B_{2i-k}(t-ih)u(t-kh) \\
&+ \sum_{i=0}^{l}\int_{t_0-ih}^{t_0}(X^*(t, \tau+ih)A_{11i}(\tau+ih) + Y^*(t, \tau+ih)A_{21i}(\tau+ih))\varphi(\tau)d\tau \\
&+ \sum_{i=0}^{l}\int_{t_0-ih}^{t_0}(X^*(t, \tau+ih)A_{12i}(\tau+ih) + Y^*(t, \tau+ih)A_{22i}(\tau+ih))\psi(\tau)d\tau \\
&+ \sum_{i=0}^{l}\int_{t_0-ih}^{t_0}(X^*(t, \tau+ih)B_{1i}(\tau+ih) + Y^*(t, \tau+ih)B_{2i}(\tau+ih))\xi(\tau)d\tau \\
&+ \sum_{k=T_t+1}^{T_t+l}\sum_{i=k-l}^{T_t} Z^*(t, t-ih)A_{21k-i}(t-ih)\varphi(t-kh) \\
&+ X^*(t, t_0+0)x_0 + \sum_{k=T_t+1}^{T_t+l}\sum_{i=k-l}^{T_t} Z^*(t, t-ih) \times A_{22k-i}(t-ih)\psi(t-kh) \\
&+ \sum_{k=T_t+1}^{T_t+l}\sum_{i=k-l}^{T_t} Z^*(t, t-ih)B_{2k-i}(t-ih)\xi(t-kh) \\
&= \begin{cases} x(t) \ \text{for} \ t \ge t_0, \ \text{if} \ X^*(t, t-0) = I_n \in \mathbf{R}^{n\times n} \ \text{and} \ Z^*(t, t) = 0 \in \mathbf{R}^{n\times m} \\ y(t) \ \text{for} \ t \ge t_0, \ \text{if} \ X^*(t, t-0) = A_{210}(t) \in \mathbf{R}^{m\times n} \ \text{and} \ Z^*(t, t) = I_m \in \mathbf{R}^{m\times m}. \end{cases}
\end{aligned} \tag{9}$$

Note that, for $x(t)$ with $t \ge t_0$, we have $Z^*(t, t-kh) = 0$, $k = 0, 1, \dots, T_t$, and it follows from (6) that $X^*(t, \tau)$ is continuous for $\tau \le t$. Therefore, the terms in (9) that involve $Z^*(t, \cdot)$ can be omitted. For stationary systems, the adjoint system and representation (9) simplify since the solution $X^*(t, \tau)$, $Y^*(t, \tau)$, $Z^*(t, \tau)$ can be chosen to be a function of the single argument $t - \tau$; thus, the usual direction of time can be used in the adjoint system: $t \ge t_0 = 0$.

The following growth estimate for solutions to hybrid systems is valid.

Theorem 2. *For each solution to stationary system* (1), (2) *corresponding to initial data* (3) *with an admissible control* $u(\cdot)$ *that increases no faster than exponentially* (*i.e.,* $\|u(t)\| \le Me^{\sigma t}$, $t \ge 0$, *where* M *and* σ *are positive constants*), *there are positive numbers* L *and* α *such that* $\|x(t)\| \le Le^{\alpha t}$ *and* $\|y(t)\| \le Le^{\alpha t}$ *for* $t \ge 0$.

An exponential estimate of solutions for $x(t)$ can be obtained following the classical approach [8, 9] designed for solutions to systems with an aftereffect, while a subtler estimation procedure is required for $y(t)$. In view of the resulting estimates, stationary systems (1)–(3) can be analyzed by applying the Laplace transform. Therefore, the solutions to such systems can be represented as series in terms of the solutions to the determining equations of these systems.

Following [2, 11], we introduce the determining equation for stationary system (1), (2):

$$\begin{aligned}
X_k(t) &= \sum_{i=0}^{l}(A_{11i}X_{k-1}(t-ih) + A_{12i}Y_{k-1}(t-ih) + B_{1i}U_{k-1}(t-ih)), \\
Y_k(t) &= \sum_{i=0}^{l}(A_{21i}X_k(t-ih) + A_{22i}Y_k(t-ih) + B_{2i}U_k(t-ih)), \\
&t \ge 0, \quad k = 0, 1, 2, \dots,
\end{aligned} \tag{10}$$

with the initial conditions $X_k(t) = 0$ and $Y_k(t) = 0$ if $k < 0$ or $t < 0$; $U_k(t) = 0$ if $k^2 + t^2 \ne 0$; and $U_0(0) = I_r$, where I_k is the $k \times k$ identity matrix.

Lemma 1. *We have the identities*

$$\left(\sum_{i=0}^{l} A_{11i}\omega^i + \sum_{i=0}^{l} A_{12i}\omega^i\left(I_m - \sum_{i=0}^{l} A_{22i}\omega^i\right)^{-1}\sum_{i=0}^{l} A_{21i}\omega^i\right)^k$$
$$\times\left(\sum_{i=0}^{l} B_{1i}\omega^i + \sum_{i=0}^{l} A_{12i}\omega^i\left(I_m - \sum_{i=0}^{l} A_{22i}\omega^i\right)^{-1}\right.$$
$$\left.\times\sum_{i=0}^{l} B_{2i}\omega^i\right) \equiv \sum_{j=0}^{+\infty} X_{k+1}(jh)\omega^j,$$

$$\left(I_m - \sum_{i=0}^{l} A_{22i}\omega^i\right)^{-1}\sum_{i=0}^{l} A_{21i}\omega^i\left(\sum_{i=0}^{l} A_{11i}\omega^i\right.$$
$$\left.+\sum_{i=0}^{l} A_{12i}\omega^i\left(I_m - \sum_{i=0}^{l} A_{22i}\omega^i\right)^{-1}\sum_{i=0}^{l} A_{21i}\omega^i\right)^k \qquad (11)$$
$$\times\left(\sum_{i=0}^{l} B_{1i}\omega^i + \sum_{i=0}^{l} A_{12i}\omega^i\left(I_m - \sum_{i=0}^{l} A_{22i}\omega^i\right)^{-1}\right.$$
$$\left.\times\sum_{i=0}^{l} B_{2i}\omega^i\right) \equiv \sum_{j=0}^{+\infty} Y_{k+1}(jh)\omega^j,$$
$$k = 0, 1, \ldots;$$
$$\left(I_m - \sum_{i=0}^{l} A_{22i}\omega^i\right)^{-1}\sum_{i=0}^{l} B_{2i}\omega^i \equiv \sum_{j=0}^{+\infty} Y_0(jh)\omega^j$$

for $|\omega| < \omega_1$, *where* ω_1 *is a sufficiently small positive number.*

The lemma is proved by induction. Identity (11) was obtained in [12] in the special case where system (1), (2) is one with a retarded argument.

Theorem 3. *The solution to stationary system* (1)–(3) *can be represented in the form of the series*

$$x(t) = \sum_{k=0}^{+\infty}\sum_{\substack{i\\(t-ih>0)}} X_{k+1}(ih)\int_0^{t-ih}\frac{(t-\tau-ih)^k}{k!}u(\tau)d\tau$$
$$+ x(t; 0, x_0, \psi, \varphi, \xi, 0), \quad t > t_0 = 0, \qquad (12)$$
$$y(t) = \sum_{k=0}^{+\infty}\sum_{\substack{i\\(t-ih>0)}} Y_{k+1}(ih)\int_0^{t-ih}\frac{(t-\tau-ih)^k}{k!}u(\tau)d\tau$$
$$+ \sum_{\substack{i\\(t-ih\ge 0)}} Y_0(ih)u(t-ih) + y(t; 0, x_0, \psi, \varphi, \xi, 0),$$
$$t \ge t_0 = 0.$$

Note that the series representations in terms of the solutions to the determining equations for the solutions $x(\cdot; 0, x_0, \psi, \varphi, \xi, 0)$ and $y(\cdot; 0, x_0, \psi, \varphi, \xi, 0)$ to homogeneous system (1)–(3) do not have such a complete form as for $x(\cdot; 0, 0, 0, 0, 0, u)$ and $y(\cdot; 0, 0, 0, 0, 0, u)$. However, if $\psi(\tau) = 0$, $\varphi(\tau) = 0$, $\xi(\tau) = 0$, and $\tau \in [-lh, 0]$, we have the expansion

$$x(t; 0, x_0, 0, 0, 0, 0)$$
$$= \sum_{k=0}^{+\infty}\sum_{\substack{i\\(t-ih\ge 0)}} X^0_{k+1}(ih)\frac{(t-ih)^k}{k!}x_0, \quad t \ge 0,$$
$$y(t; 0, x_0, 0, 0, 0, 0)$$
$$= \sum_{k=0}^{+\infty}\sum_{\substack{i\\(t-ih\ge 0)}} Y^0_{k+1}(ih)\frac{(t-ih)^k}{k!}x_0, \quad t \ge 0,$$

where $0^0 = 1$ and $X^0_k(t)$ and $Y^0_k(t)$ are the solution to Eq. (10) at $r = n$ with the matrices $B_{10} = I_n$, $B_{1i} = 0$, $B_{20} = 0$, and $B_{2i} = 0$ $(i = 0, 1, \ldots, l)$.

Let H be an arbitrary $p \times (n+m)$ matrix.

Definition 1. System (1)–(3) is called H–t_1-controllable for $t_1 > 0$ if, for any vector $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in \mathbf{R}^{n+m}$ and any initial data, there exists a piecewise continuous control $u(\cdot)$ such that the solution to the system has the property $H\begin{bmatrix} x(t_1) \\ y(t_1) \end{bmatrix} = H\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$.

As a consequence of Theorem 3, we obtain the following criterion for relative H–t_1-controllability.

Proposition 1. *Stationary system* (1)–(3) *is* H–t_1-*controllable if and only if*

$$\operatorname{rank}\left[H, H\begin{bmatrix} X_k(t) \\ Y_k(t) \end{bmatrix}, \quad t \in [0, t_1), \quad H\begin{bmatrix} 0 \\ Y_0(jh) \end{bmatrix},\right.$$
$$\left. j = 1, 2, \ldots, T_{t_1}; \quad k = 1, 2, \ldots, +\infty\right]$$
$$= \operatorname{rank}\left[H\begin{bmatrix} X_k(t) \\ Y_k(t) \end{bmatrix}, \quad t \in [0, t_1), \quad H\begin{bmatrix} 0 \\ Y_0(jh) \end{bmatrix},\right.$$
$$\left. j = 1, 2, \ldots, T_{t_1}; \quad k = 1, 2, \ldots, +\infty\right].$$

In Proposition 1, we have to check the rank of a matrix containing an infinite number of columns. Below, additional properties of solutions to the determining equations are established that reduce the solvability of the relative controllability problem for system (1), (2) to the determination of the rank of a matrix with a finite number of rows and columns.

Let the numbers $r_{00} = 1$ and r_{ij} ($i = 0, 1, \ldots, n$; $j = 0, 1, \ldots, n(m+1)l$) be determined by the (characteristic) equation

$$0 = \left(\det\left(I_m - \sum_{i=0}^{l} A_{22i}\omega^i\right)^n\right)\det\left(\lambda I_n - \sum_{i=0}^{l} A_{11i}\omega^i - \sum_{i=0}^{l} A_{12i}\omega^i\left(I_m - \sum_{i=0}^{l} A_{22i}\omega^i\right)^{-1}\sum_{i=0}^{l} A_{21i}\omega^i\right) = \sum_{i=0}^{n}\sum_{j=0}^{n(m+1)l} r_{ij}\lambda^{n-i}\omega^j. \quad (13)$$

Lemma 2. *The solutions $X_\gamma(t)$ and $Y_\gamma(t)$ ($t \ge 0$) to determining equation* (10) *satisfy characteristic equation* (13):

$$\begin{bmatrix} X_\gamma(kh) \\ Y_\gamma(kh) \end{bmatrix} = -\sum_{j=1}^{\theta_k} r_{0j}\begin{bmatrix} X_\gamma((k-j)h) \\ Y_\gamma((k-j)h) \end{bmatrix} - \sum_{i=1}^{n}\sum_{j=0}^{\theta_k} r_{ij}\begin{bmatrix} X_{\gamma-i}((k-j)h) \\ Y_{\gamma-i}((k-j)h) \end{bmatrix},$$

where $\theta_k = \min\{k, n(m+1)l\}$ *for* $k = 0, 1, \ldots$ *and* $\gamma = n + 1, n + 2, \ldots$.

Lemma 3. *There are real numbers p_{ij} ($i = 0, 1, \ldots, (m+n)l$ and $j = 0, 1, \ldots, \theta_k = \min\{k, n(n+m)l^2\}$, where $k = 0, 1, \ldots$) such that*

$$\begin{bmatrix} X_k(\gamma h) \\ Y_k(\gamma h) \end{bmatrix} = -\sum_{j=1}^{\theta_k} p_{0j}\begin{bmatrix} X_{k-j}(\gamma h) \\ Y_{k-j}(\gamma h) \end{bmatrix} - \sum_{i=1}^{(m+n)l}\sum_{j=0}^{\theta_k} p_{ij}\begin{bmatrix} X_{k-j}((\gamma-i)h) \\ Y_{k-j}((\gamma-i)h) \end{bmatrix},$$

where $\gamma \ge (n+m)l^2 + 1$ *and* $\gamma \in N$.

Lemmas 2 and 3 are proved by induction. Similar lemmas for the special case of system (1), (2) with no difference equation were derived in [12].

Lemmas 2 and 3 can be used to improve the H–t_1-controllability criterion.

Theorem 4. *Stationary system* (1)–(3) *is H–t_1-controllable if and only if*

$$\operatorname{rank}\left[H, H\begin{bmatrix} X_k(jh) \\ Y_k(jh) \end{bmatrix}, H\begin{bmatrix} 0 \\ Y_0(ih) \end{bmatrix}, \begin{array}{c} k = 0, 1, \ldots, n;\ j = 0, 1, \ldots, \min\{T_{t_1-0}, (m+n)l\}; \\ i = 0, 1, \ldots, \min\{T_{t_1}, (m+n)l\} \end{array}\right]$$
$$= \operatorname{rank}\left[H\begin{bmatrix} X_k(jh) \\ Y_k(jh) \end{bmatrix}, H\begin{bmatrix} 0 \\ Y_0(ih) \end{bmatrix}, \begin{array}{c} k = 0, 1, \ldots, n;\ j = 0, 1, \ldots, \min\{T_{t_1-0}, (m+n)l\}; \\ i = 0, 1, \ldots, \min\{T_{t_1}, (m+n)l\} \end{array}\right],$$

where $T_{t_1-0} = \lim_{\varepsilon\to+0} T_{t_1-\varepsilon}$.

Thus, we can say that the property of relative H–t_1-controllability becomes saturated.

Proposition 2. *Stationary system* (1)–(3) *is H–t_1-controllable for $t_1 > (m+n)l = t_*$ if and only if it is H–t_*-controllable.*

For special systems (1), (2), the problem of relative t_1-controllability was studied in [2, 3]; however, the method of [2, 3] supposed determining the rank of a matrix whose size depends on t_1 and unboundedly increases with t_1.

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