V. Conclusion

In this note, a new approach has been established to study the problem of stochastic stability for a class of nonlinear stochastic systems with semi-Markovian jump parameters. It has been shown that the existing results on stochastic stability for Markovian jump systems also hold for semi-Markovian jump systems. The semi-Markovian jump systems are less conservative and more applicable in real practices. A numerical example is given to illustrate the feasibility and effectiveness of the theoretic results obtained.

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On the Observability of Linear Differential-Algebraic Systems With Delays

V. M. Marchenko, O. N. Poddubnaya, and Z. Zaczkiewicz

***Abstract*—The problem of $\mathbb{R}^n$-observability is considered for the simplest linear time-delay differential-algebraic system consisting of differential and difference equations. A determining equation system is introduced and a number of algebraic properties of the determining equation solutions is established, in particular, the well-known Hamilton–Cayley matrix theorem is generalized to the solutions of determining equation. As a result, an effective parametric rank criterion for the $\mathbb{R}^n$-observability is given. A dual controllability result is also formulated.**

***Index Terms*—Determining equations, differential-algebraic systems, duality, observability, time-delay.**

I. Introduction

The note deals with linear stationary differential-algebraic systems with delays (DAD systems), with some equations being differential, the other—difference, with some variables being continuous the other—piecewise continuous (see also [1]–[5]). Observe that some kinds of neutral type time-delay and discrete-continuous hybrid systems can be regarded as examples of DAD systems.

Example 1: Consider a linear neutral type time-delay system

$$\frac{d}{dt}\left(y(t) - A_{22}y(t-h)\right) = A_{11}y(t) + A_{12}y(t-h). \qquad (1)$$

If we denote $x(t) = y(t) - A_{22}y(t-h)$, we obtain the following DAD system:

$$\dot{x}(t) = A_{11}x(t) + (A_{11}A_{22} + A_{12})y(t-h)$$
$$y(t) = x(t) + A_{22}y(t-h).$$

Example 2: Consider the following linear discrete-continuous system:

$$\dot{x}(t) = A_{11}x(t) + A_{12}y[k], \quad t \in [kh, (k+1)h) \qquad (2a)$$
$$y[k] = A_{21}x(kh) + A_{22}y[k-1], \qquad k = 0, 1, \ldots \qquad (2b)$$

with initial conditions

$$x(0) = x(0+) = x_0 \quad y[-1] = y_0,$$

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where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$, and A_{11}, A_{12}, A_{21}, A_{22} are constant matrices of compatible sizes. Consider

$$\tilde{y}(t) = \begin{bmatrix} x(kh) \\ y[k] \end{bmatrix}, \text{ for } t \in [kh, (k+1)h), \qquad k = 0, 1, \ldots$$

where

$$\begin{aligned} x(kh) &= e^{A_{11}(kh-(k-1)h)} x(kh-h) \\ &\quad + \int_{kh-h}^{kh} e^{A_{11}(kh-\tau)} A_{12} y[k-1] d\tau \\ &= e^{A_{11}h} x(kh-h) \\ &\quad + \int_0^h e^{A_{11}(h-s)} ds A_{12} y[k-1], \qquad k = 0, 1, \ldots \end{aligned}$$

and initial conditions are given by

$$\begin{aligned} x(0) &= x(0+) = x_0 \\ \tilde{y}(\tau) &= \begin{bmatrix} e^{-A_{11}h}\left(x_0 - \int_0^h e^{A_{11}(h-\tau)} A_{12} y_0 d\tau\right) \\ y_0 \end{bmatrix}, \ \tau \in [-h, 0). \end{aligned}$$

It is not difficult to see that (2) can be represented as a DAD system of the form

$$\begin{aligned} \dot{x}(t) &= \tilde{A}_{11} x(t) + \tilde{A}_{12} \tilde{y}(t) \\ \tilde{y}(t) &= \tilde{A}_{21} x(t) + \tilde{A}_{22} \tilde{y}(t-h), \qquad t \geq 0 \end{aligned}$$

with $\tilde{A}_{11} = A_{11}$, $\tilde{A}_{12} = [0 \ A_{12}]$, $\tilde{A}_{21} = 0$

$$\tilde{A}_{22} = \begin{bmatrix} e^{A_{11}h} & \int_0^h e^{A_{11}(h-\tau)} A_{12} d\tau \\ A_{21} e^{A_{11}h} & A_{22} + A_{21} \int_0^h e^{A_{11}(h-\tau)} A_{12} d\tau \end{bmatrix}.$$

We believe that the previous examples provide the motivation for further investigation of differential-algebraic systems with delays

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{l} (A_{11i} x(t-h_i) + A_{12i} y(t-h_i)) \\ y(t) &= \sum_{i=0}^{l} (A_{21i} x(t-h_i) + A_{22i} y(t-h_i)) \end{aligned}$$

where $A_{11i} \in \mathbb{R}^{n\times n}$, $A_{12i} \in \mathbb{R}^{n\times m}$, $A_{21i} \in \mathbb{R}^{m\times n}$, $A_{22i} \in \mathbb{R}^{m\times m}$, $A_{220} = 0$, and $0 < h_0 < h_1 < \ldots < h_l$ are constant delays.

The problem of controllability of systems with after-effect began its history with [6], where the problem of controllability to zero function (complete controllability) was formulated for the simplest retarded type system. Simultaneously, Kirillova and Churakova [7] and, independently, Weiss [8] investigated the problem of relative (Euclidean, $\mathbb{R}^n$-) controllability. For such a type of controllability, effective rank conditions were obtained [7] in the terms of determining equations. Later, the determining equation techniques were extended to the problems of $\mathbb{R}^n$-controllability and observability for various classes of linear stationary systems with several concentrated delays and to neutral time-delay systems (see, for example, [2], [9]–[14], and the references therein). The book [11] (see also [13]) and survey [10] present a general overview of determining equation techniques.

In this note, we consider DAD systems of the simplest form. In order to investigate observability of such a system, we introduce determining equations that describe rank type conditions for $\mathbb{R}^n$-observability with respect to the continuous variable. The rank type conditions are used to establish a $\mathbb{R}^n$-observability–controllability duality principle for the DAD systems.

II. Preliminaries

In this section, we extend the well-known ordinary time-delay determining equation techniques [10], [11] to the investigation of DAD systems. Let us consider observation system

$$\dot{x}(t) = A_{11} x(t) + A_{12} y(t), \ \ t > 0 \tag{3a}$$
$$y(t) = A_{21} x(t) + A_{22} y(t-h), \qquad t \geq 0 \tag{3b}$$

with output

$$z(t) = B_1 x(t) + B_2 y(t), \tag{3c}$$

and initial conditions

$$x(+0) = x_0, \ \ y(\tau) = \psi(\tau), \qquad \tau \in [-h, 0), \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^r$, $t \geq 0$; $A_{11} \in \mathbb{R}^{n\times n}$, $A_{12} \in \mathbb{R}^{n\times m}$, $A_{21} \in \mathbb{R}^{m\times n}$, $A_{22} \in \mathbb{R}^{m\times m}$, $B_1 \in \mathbb{R}^{r\times n}$, $B_2 \in \mathbb{R}^{r\times m}$; $0 < h$ is a constant delay; $x_0 \in \mathbb{R}^n$; $\psi \in PC([-h, 0), \mathbb{R}^m)$, and $PC([-h, 0), \mathbb{R}^m)$ is a set of piecewise continuous m-vector-functions in $[-h, 0]$. Observe that $y(t)$ at $t = 0$ is determined by (3b).

Using the Laplace transformation, one can prove (details are in [15]) that the solution of (3) and (4) can be represented as follows:

$$\begin{aligned} x(t) &= \sum_{k=0}^{+\infty} \sum_{\substack{i,j \\ t-(j+i)h>0}} X_{k+1}(jh) A_{12} (A_{22})^{i+1} \\ &\quad \times \int_0^{t-(j+i)h} \frac{(t-(j+i)h-\tau)^k}{k!} \psi(\tau - h) d\tau \\ &\quad + \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^k}{k!} X_{k+1}(jh) x_0 \\ y(t) &= \sum_{k=0}^{+\infty} \sum_{\substack{i,j \\ t-(j+i)h>0}} Y_{k+1}(jh) A_{12} (A_{22})^{i+1} \\ &\quad \times \int_0^{t-(j+i)h} \frac{(t-(j+i)h-\tau)^k}{k!} \psi(\tau - h) d\tau \\ &\quad + \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^k}{k!} Y_{k+1}(jh) x_0 \\ &\quad + \sum_{i=0}^{+\infty} (A_{22})^{i+1} \psi\left(t-(i+1)h\right) \end{aligned}$$

where $\psi(\tau) \equiv 0$ for $\tau \notin [-h, 0)$ and functional matrices $X_k(t)$, $Y_k(t), t \geq 0, k = 0, 1, \ldots$, satisfy the following determining equations of (3):

$$X_k(t) = A_{11} X_{k-1}(t) + A_{12} Y_{k-1}(t) + U_{k-1}(t) \tag{5a}$$
$$Y_k(t) = A_{21} X_k(t) + A_{22} Y_k(t-h) \tag{5b}$$
$$Z_k(t) = B_1 X_k(t) + B_2 Y_k(t), \qquad t \geq 0, \ \ k = 0, 1, 2, \ldots \tag{5c}$$

with initial conditions

$$X_k(t)=0,\ \ Y_k(t)=0,\ \ Z_k(t)=0 \ \text{ for } \ t<0 \ \text{ or } \ k\le 0$$
$$U_0(0)=I_n,\ \ U_k(t)=0 \ \text{ for } \ t^2+k^2\ne 0.$$

The previous equations are introduced in accordance with the standard determining equation techniques [7], [10], [11] (see also [2], [13], and [14]). It is not difficult to see that $X_k(t)=0, Y_k(t)=0, Z_k(t)=0$ for $t\ne jh$, where $j=0,1,\dots$ and $k=0,1,\dots$

Here, we establish some algebraic properties of $Z_k(t)$.

Lemma 1: The following identity holds:

$$\left(B_1+B_2(I_m-A_{22}\omega)^{-1}A_{21}\right)\left(A_{11}+A_{12}(I_m-A_{22}\omega)^{-1}A_{21}\right)^i \equiv \sum_{l=0}^{+\infty} Z_{i+1}(lh)\omega^l, \qquad i=0,1,\dots \tag{6}$$

where $|\omega|<\omega_1$ and ω_1 is a sufficiently small real number.

Proof: See the Appendix. □

Let us define

$$A(\omega)=A_{11}+A_{12}(I_m-A_{22}\omega)^{-1}A_{21}\in\mathbb{R}^{n\times n}(\omega)$$
$$C(\omega)=\left(B_1+B_2(I_m-A_{22}\omega)^{-1}A_{21}\right)\in\mathbb{R}^{r\times n}(\omega).$$

Here and in what follows, $\mathbb{R}^{p\times q}(\omega)$ and $\mathbb{R}^{p\times q}[\omega]$ are the sets of p by q matrices with rational and polynomial entries in ω, respectively.

The characteristic equation of $A(\omega)$ is given by

$$\begin{aligned}0=\Delta(\lambda)&=\det\left(\lambda I_n-A_{11}-A_{12}(I_m-A_{22}\omega)^{-1}A_{21}\right)\\ &=\frac{1}{(\alpha(\omega))^n}\det\left(\lambda\alpha(\omega)I_n-\alpha(\omega)A_{11}-A_{12}Q_1(\omega)A_{21}\right)\\ &=\frac{1}{(\alpha(\omega))^n}\sum_{i=0}^{n}\sum_{j=0}^{nm} r_{ij}\lambda^{n-i}\omega^j=0\end{aligned} \tag{7}$$

where $Q_1(\omega)\in\mathbb{R}^{m\times m}[\omega]$ is the adjoint of $(I_m-A_{22}\omega)$, $\det(I_m-A_{22}\omega)=\alpha(\omega)\in\mathbb{R}^{1\times 1}[\omega]$, real numbers r_{ij}, $i=0,1,\dots,n$; $j=0,1,\dots,nm$, are defined by elements of matrices $A_{11}, A_{12}, A_{21}, A_{22}$, and $r_{00}=1$.

Let us rewrite identity (7) as follows:

$$\lambda^n=-\sum_{j=1}^{nm} r_{0j}\lambda^n\omega^j-\sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}\lambda^{n-i}\omega^j. \tag{8}$$

Then, we can formulate the following.

Lemma 2: The solutions $Z_k(t), t\ge 0$, of the determining equation (5c) satisfy the condition

$$Z_k(lh)=-\sum_{j=1}^{\theta_l} r_{0j}Z_k((l-j)h)-\sum_{i=1}^{n}\sum_{j=0}^{\theta_l} r_{ij}Z_{k-i}((l-j)h)$$

for $l=0,1,\dots$, where $\theta_l=\min\{l,nm\}$ and $k=n+1,n+2,\dots$.

Proof: See the Appendix. □

Similar to Lemmas 1 and 2, we can formulate Lemmas 3 and 4.

Lemma 3: The following identities hold:

$$\begin{aligned}&\left(B_1(I_n-A_{11}\omega)^{-1}A_{12}\omega+B_2\right)\\ &\times\left(\left(I_m-A_{21}(I_n-A_{11}\omega)^{-1}A_{12}\omega\right)^{-1}A_{22}\right)^l\\ &\times\left(A_{21}\left(I_n-(A_{11}+A_{12}A_{21})\omega\right)^{-1}\right)\\ &\equiv\sum_{k=1}^{+\infty} Z_k(lh)\omega^{k-1}, \qquad l=1,2,\dots\end{aligned}$$

where $|\omega|<\omega_1$ and ω_1 is a sufficiently small real number.

Let us introduce the following notation:

$$\begin{aligned}D(\omega)&=\left(I_m-A_{21}(I_n-A_{11}\omega)^{-1}A_{12}\omega\right)^{-1}A_{22}\in\mathbb{R}^{m\times m}(\omega)\\ F(\omega)&=\left(A_{21}\left(I_n-(A_{11}+A_{12}A_{21})\omega\right)^{-1}\right)\in\mathbb{R}^{m\times n}(\omega)\\ G(\omega)&=\left(B_1(I_n-A_{11}\omega)^{-1}A_{12}\omega+B_2\right)\in\mathbb{R}^{r\times m}(\omega)\\ \beta(\omega)&=\det(I_n-A_{11}\omega)\\ \mu(\omega)&=\det\left(I_m\beta(\omega)-A_{21}Q_2(\omega)A_{12}\omega\right)\end{aligned}$$

$Q_2(\omega)\in\mathbb{R}^{n\times n}[\omega]$ and $Q_3(\omega)\in\mathbb{R}^{m\times m}[\omega]$ denote the adjoints of $(I_n-A_{11}\omega)$ and $(I_m\beta(\omega)-A_{21}Q_2(\omega)A_{12}\omega)$ respectively.

We transform the characteristic equation of $D(\omega)$, $\Delta(\lambda)=\det(\lambda I_m-D(\omega))=0$, as follows:

$$\begin{aligned}0&=\det\left(\lambda I_m-\left(I_m-A_{21}\frac{Q_2(\omega)}{\beta(\omega)}A_{12}\omega\right)^{-1}A_{22}\right)\\ &=\det\left(\lambda I_m-\beta(\omega)\left(I_m\beta(\omega)-A_{21}Q_2(\omega)A_{12}\omega\right)^{-1}A_{22}\right)\\ &=\frac{1}{\mu(\omega)^m}\det\left(\lambda\mu(\omega)I_m-\beta(\omega)Q_3(\omega)A_{22}\right)\end{aligned}$$

which, when $|\omega|<\omega_1$ and ω_1 is a sufficiently small positive number, is equivalent to

$$0=\det\left(\lambda\mu(\omega)I_m-\beta(\omega)Q_3(\omega)A_{22}\right)=\sum_{i=0}^{m}\sum_{j=0}^{nm^2} p_{ij}\lambda^{m-i}\omega^j \tag{9}$$

where p_{ij}, $i=0,1,\dots,m$; $j=0,1,\dots,nm^2$, are real numbers expressed by elements of matrices $A_{11}, A_{12}, A_{21}, A_{22}$, and $p_{00}=1$.

We can now formulate the following.

Lemma 4: Solutions $Z_k(lh), k\ge 1, l\ge 0$, of determining equation (5c) satisfy the following conditions:

$$Z_k(lh)=-\sum_{j=1}^{\widetilde{\theta_k}} p_{0j}Z_{k-j}(lh)-\sum_{i=1}^{m}\sum_{j=0}^{\widetilde{\theta_k}} p_{ij}Z_{k-j}((l-i)h)$$

where $k=1,2,\dots$, $l=m+1,m+2,\dots$, and $\widetilde{\theta_k}=\min\{k-1,nm^2\}$.

Lemmas 2 and 4 are generalizations of the Hamilton–Cayley matrix theorem to solution $Z_k(t)$ of determining equation (5c).

We can prove the following.

Lemma 5: Functions $f_{kj}(t)=(t-jh)^k/k!$ for $t-jh\ge 0$ and $f_{kj}(t)=0$ for $t-jh<0$, where $k=0,1,\dots$; $j=0,1,\dots$, are linearly independent for $t\ge 0$.

Proof: For $t\ge 0$, $t\in[jh,(j+1)h)$, $j=0$, assume that $\sum_{k=0}^{+\infty}\alpha_{k0}(t^k/k!)\equiv 0$, $t\in[0,h)$, $\alpha_{ij}\in\mathbb{R}$. By letting $t=0$, we obtain $\alpha_{00}=0$. This implies $\sum_{k=1}^{+\infty}\alpha_{k0}(t^{k-1}/k!)\equiv 0$, $t\in[0,h)$, and $\alpha_{10}=0$. Analogously, $\alpha_{l0}=0$, $l=0,1,\dots$ Hence, Lemma 5 holds true for $j=0$. Then, the proof is by induction on j. □

III. Main Results

A. Criterion for $\mathbb{R}^n$-Observability of Differential-Algebraic Systems With Delays

Let $x(t,\psi,x_0)$, $y(t,\psi,x_0)$ be the solution at time $t \geq 0$ of (3) corresponding to initial conditions (4). Similarly, $z(t)=z(t,\psi,x_0)$, $\tilde{z}(t)=\tilde{z}(t,\psi,\tilde{x}_0)$ denote the outputs corresponding to the solutions $x(t)=x(t,\psi,x_0)$, $y(t)=y(t,\psi,x_0)$ and $\tilde{x}(t)=\tilde{x}(t,\psi,\tilde{x}_0)$, $\tilde{y}(t)=\tilde{y}(t,\psi,\tilde{x}_0)$, respectively.

Definition 1: System (3) is said to be $\mathbb{R}^n$-observable with respect to x if for every x_0, $\tilde{x}_0 \in \mathbb{R}^n$ the condition

$$z(t,\psi,x_0) \equiv \tilde{z}(t,\psi,\tilde{x}_0), \text{ for every } \psi \in PC([-h,0),\mathbb{R}^m), \text{ and for } t \geq 0$$

implies that $x_0=\tilde{x}_0$.

Theorem 1: System (3) is $\mathbb{R}^n$-observable with respect to x if and only if

$$\operatorname{rank}\begin{bmatrix} Z_\eta(\xi h) \\ \xi=0,\dots,m;\ \eta=1,\dots,n \end{bmatrix} := \operatorname{rank}\begin{bmatrix} Z_1(0) \\ Z_1(h) \\ \vdots \\ Z_1(mh) \\ Z_2(0) \\ \vdots \\ Z_n(mh) \end{bmatrix} = n.$$

Proof: By the series representation of the solutions $x(t), y(t)$ and (3c), $z(t,\phi,x_0)=\tilde{z}(t,\phi,\tilde{x}_0)$ is equivalent to the following:

$$\begin{aligned} &B_1 \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^k}{k!} X_{k+1}(jh)x_0 \\ &\quad + B_2 \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^k}{k!} Y_{k+1}(jh)x_0 \\ &\quad = B_1 \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^k}{k!} X_{k+1}(jh)\tilde{x}_0 \\ &\quad + B_2 \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^k}{k!} Y_{k+1}(jh)\tilde{x}_0. \end{aligned}$$

It follows from here that

$$\begin{aligned} &\sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^k}{k!} [B_1, B_2] \begin{bmatrix} X_{k+1}(jh) \\ Y_{k+1}(jh) \end{bmatrix} (x_0-\tilde{x}_0) \\ &= \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^k}{k!} Z_{k+1}(jh)(x_0-\tilde{x}_0) \\ &= 0. \end{aligned}$$

By Lemma 5, we conclude that the following linear system of algebraic equations has only trivial solution:

$$W_\infty^\infty(x_0-\tilde{x}_0)=0 \tag{10}$$

where

$$W_k^l = \begin{bmatrix} Z_\eta(\xi h), \\ \eta=1,\dots,k;\ \xi=0,\dots,l \end{bmatrix}.$$

By Lemma 2, $Z_k(lh)$ for $k>n$ is a linear combination of $Z_\eta(\xi h)$ for $\eta=1,2\dots,n;\ \xi=0,1\dots$. From the above, taking into account Lemma 4, it is easy to see that $Z_k(lh)$, where $k>n, l>m$, are linear combinations of $Z_\eta(\xi h)$, $\eta=1,2\dots,n;\ \xi=0,1\dots,m$. Thus

$$\operatorname{rank} W_\infty^\infty = \operatorname{rank} W_n^m.$$

Combining these with (10), we complete the proof. □

B. Duality

Let us consider a dual control system

$$\dot{x}^*(t)=A_{11}^T x^*(t)+A_{21}^T y^*(t)+B_1^T u(t),\ t>0 \tag{11a}$$

$$y^*(t)=A_{12}^T x^*(t)+A_{22}^T y^*(t-h)+B_2^T u(t),\qquad t\geq 0 \tag{11b}$$

with initial conditions

$$x^*(+0)=x_0^*,\ \ y^*(\tau)=\psi^*(\tau),\ \ \tau\in[-h,0)$$

where $x^*(t)\in\mathbb{R}^n$, $y^*(t)\in\mathbb{R}^m$, $u(t)\in\mathbb{R}^r$, $t\geq 0$, $x_0^*\in\mathbb{R}^n$; $\psi^*\in PC([-h,0),\mathbb{R}^m)$; symbol $()^T$ means transposition.

Let us consider determining equations

$$\begin{aligned} X_k^*(t) &= A_{11}^T X_{k-1}^*(t) + A_{21}^T Y_{k-1}^*(t) + B_1^T U_{k-1}^*(t) \\ Y_k^*(t) &= A_{12}^T X_k^*(t) + A_{22}^T Y_k^*(t-h) + B_2^T U_k^*(t) \\ & t\geq 0, k=0,1,\dots \end{aligned}$$

of system (11) with the following initial conditions:

$$\begin{aligned} &X_k^*(t)=0,\ \ Y_k^*(t)=0 \ \text{ if } k<0 \text{ or } t<0 \\ &U_0^*(0)=I_r,\ \ U_k^*(t)=0 \ \text{ if } t^2+k^2\neq 0. \end{aligned}$$

Definition 2: System (11) is said to be $\mathbb{R}^n$-controllable with respect to x^* if for any initial data x_0^*, ψ^* and any $x_*^*\in\mathbb{R}^n$ there exist a time moment $t_*>0$ and a piecewise continuous control $u(t)$, $t\in[0,t_*]$, such that for the corresponding solution $x^*(t)=x^*(t,x_0^*,\psi^*,u)$, $t>0$, the condition $x^*(t_*)=x_*^*$ is valid.

The following two statements hold [14].

Proposition 1: We have:

$$\begin{aligned} &\left(A_{11}^T + A_{21}^T\left(I_m - A_{22}^T\omega\right)^{-1} A_{12}^T\right)^k \\ &\times \left(B_1^T + A_{21}^T\left(I_m - A_{22}^T\omega\right)^{-1} B_2^T\right) \\ &\equiv \sum_{l=0}^{+\infty} X_{k+1}^*(lh)\omega^l,\ \ k=0,1,\dots \end{aligned}$$

where $|\omega|<\omega_1$ and ω_1 is a sufficiently small real number.

Proposition 1: System (11) is $\mathbb{R}^n$-controllable with respect to x^* if and only if

$$\operatorname{rank}\left[X_\eta^*(\xi h),\ \xi=0,\dots,m;\ \eta=1,\dots,n\right]=n$$

where by the symbol $[X_\eta^*(\xi h), \xi=0,\dots,m; \eta=1,\dots,n]$ we denote a block matrix of columns $X_\eta^*(\xi h), \xi=0,\dots,m; \eta=1,\dots,n$.

Now, we can state the duality result.

Theorem 2: System (3) is $\mathbb{R}^n$-observable with respect to x if and only if (11) is $\mathbb{R}^n$-controllable with respect to x^*.

Proof: By Lemma 1 and Proposition 1, we have

$$\begin{aligned}&\left(B_1 + B_2(I_m - A_{22}\omega)^{-1}A_{21}\right)\\&\quad\times\left(A_{11} + A_{12}(I_m - A_{22}\omega)^{-1}A_{21}\right)^k\\&\quad\equiv\sum_{l=0}^{+\infty} Z_{k+1}(lh)\omega^l,\ \ k=0,1,\ldots\\&\left(A_{11}^T + A_{21}^T\left(I_m - A_{22}^T\omega\right)^{-1}A_{12}^T\right)^k\\&\quad\times\left(B_1^T + A_{21}^T\left(I_m - A_{22}^T\omega\right)^{-1}B_2^T\right)\\&\quad\equiv\sum_{l=0}^{+\infty} X_{k+1}^*(lh)\omega^l,\qquad k=0,1,\ldots\end{aligned}\tag{12}$$

Transposing (12), we have

$$\sum_{l=0}^{+\infty} Z_{k+1}(lh)\omega^l = \sum_{l=0}^{+\infty} {X_{k+1}^*}^T(lh)\omega^l.$$

Then, comparing coefficients of the same power of ω, we have

$$Z_k(lh) = X_k^{*T}(lh)$$

for $k=0,1,\ldots$ and $l=0,1,\ldots$ It follows that

$$\begin{bmatrix} Z_\eta(\xi h)\\ \xi=0,\ldots,m;\ \eta=1,\ldots,n\end{bmatrix} = \left[X_\eta^*(\xi h), \xi=0,1,\ldots,m;\ \eta=1,2,\ldots,n\right]^T$$

which completes the proof. □

IV. Conclusion

In this note, we have considered the simplest stationary linear differential-algebraic systems of observation and control with delays. For such systems, a number of algebraic properties of determining equation have been established in order to obtain an effective rank condition for $\mathbb{R}^n$-observability in terms of determining equation solutions and, as a result, the "observability-controllability" duality principle has been proposed. The results obtained can be generalized to differential-algebraic systems with several state and control delays and to problems of functional observability and controllability. A more general "observability-controllability" duality principle can also be formulated for such problems. This will be the object of another note.

Appendix

A. Proof of Lemma 1

Multiplying the (5b) by ω^j at $t=jh$ and summing over j from 0 to $+\infty$, we obtain

$$\begin{aligned}\sum_{j=0}^{+\infty} Y_k(jh)\omega^j &= \sum_{j=0}^{+\infty} A_{21}X_k(jh)\omega^j + \sum_{j=0}^{+\infty} A_{22}Y_k\left((j-1)h\right)\omega^j\\&= \sum_{j=0}^{+\infty} A_{21}X_k(jh)\omega^j + \sum_{j=-1}^{+\infty} A_{22}Y_k(jh)\omega^{j+1}.\end{aligned}$$

Hence, we have

$$\sum_{j=0}^{+\infty} Y_k(jh)\omega^j = (I_m - A_{22}\omega)^{-1}A_{21}\sum_{j=0}^{+\infty} X_k(jh)\omega^j.\tag{13}$$

Then, we obtain

$$\begin{aligned}\sum_{j=0}^{+\infty} Z_k(jh)\omega^j &= \sum_{j=0}^{+\infty} B_1X_k(jh)\omega^j + \sum_{j=0}^{+\infty} B_2Y_k(jh)\omega^j\\&= \left(B_1 + B_2(I_m - A_{22}\omega)^{-1}A_{21}\right)\\&\quad\times\sum_{j=0}^{+\infty} X_k(jh)\omega^j.\end{aligned}\tag{14}$$

It is easy to see that (6) is true for $i=0$. For $k=2, t=jh>0$, one can multiply (5a) by ω^j and sum over j from 0 to $+\infty$. Then, we have

$$\begin{aligned}\sum_{j=0}^{+\infty} X_2(jh)\omega^j &= \sum_{j=0}^{+\infty} A_{11}X_1(jh)\omega^j + \sum_{j=0}^{+\infty} A_{12}Y_1(jh)\omega^j\\&= A_{11} + \sum_{j=0}^{+\infty} A_{12}(A_{22})^jA_{21}\omega^j\\&= A_{11} + A_{12}(I_m - A_{22}\omega)^{-1}A_{21}\end{aligned}$$

where $|\omega| \le \omega_1 < (1/\|A_{22}\|)$, and (6) is true for $i=1$.

Assuming that (6) holds for $i=0,1,\ldots,p-1$, let us prove it holds true for $i=p$, i.e.,

$$\left(B_1 + B_2(I_m - A_{22}\omega)^{-1}A_{21}\right)\times\left(A_{11} + A_{12}(I_m - A_{22}\omega)^{-1}A_{21}\right)^p \equiv \sum_{l=0}^{+\infty} Z_{p+1}(lh)\omega^l$$

where p is a natural number.

Indeed, by (5a), for $k=p+1$, we obtain

$$\sum_{j=0}^{+\infty} X_{p+1}(jh)\omega^j = A_{11}\sum_{j=0}^{+\infty} X_p(jh)\omega^j + A_{12}\sum_{j=0}^{+\infty} Y_p(jh)\omega^j.$$

By (13), we have

$$\begin{aligned}&\sum_{j=0}^{+\infty} X_{p+1}(jh)\omega^j\\&\quad= A_{11}\sum_{j=0}^{+\infty} X_p(jh)\omega^j + A_{12}(I_m - A_{22}\omega)^{-1}A_{21}\\&\qquad\times\sum_{j=0}^{+\infty} X_p(jh)\omega^j\\&\quad= \left(A_{11} + A_{12}(I_m - A_{22}\omega)^{-1}A_{21}\right)\\&\qquad\times\sum_{j=0}^{+\infty} X_p(jh)\omega^j\\&\quad= \left(A_{11} + A_{12}(I_m - A_{22}\omega)^{-1}A_{21}\right)^p.\end{aligned}$$

By (14), the proof is complete.

B. Proof of Lemma 2

By the Cayley–Hamilton theorem, we have

$$(A(\omega))^n = -\sum_{j=1}^{nm} r_{0j}\,(A(\omega))^n\,\omega^j - \sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}\,(A(\omega))^{n-i}\,\omega^j,\ |\omega| < \omega_1.$$

Postmultiplying both sides by $A(\omega)^{\beta-1}$, $\beta \in \mathbb{N}$, and premultiplying by $C(\omega)$ yields

$$C(\omega)\,(A(\omega))^{n+\beta-1} = -\sum_{j=1}^{nm} r_{0j}C(\omega)\,(A(\omega))^{n+\beta-1}\,\omega^j - \sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}C(\omega)\,(A(\omega))^{n-i+\beta-1}\,\omega^j$$

and taking into account (6), we obtain

$$\sum_{l=0}^{+\infty} Z_{n+\beta}(lh)\omega^l = -\sum_{j=1}^{nm} r_{0j}\sum_{l=0}^{+\infty} Z_{n+\beta}(lh)\omega^l\omega^j - \sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}\sum_{l=0}^{+\infty} Z_{n+\beta-i}(lh)\omega^l\omega^j.$$

By the substitution $n+\beta=\gamma$, we obtain

$$\sum_{l=0}^{+\infty} Z_{\gamma}(lh)\omega^l = -\sum_{j=1}^{nm} r_{0j}\sum_{l=0}^{+\infty} Z_{\gamma}(lh)\omega^{l+j} - \sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}\sum_{l=0}^{+\infty} Z_{\gamma-i}(lh)\omega^{l+j}.$$

By letting $l+j=s(l=s-j\geq 0)$, we obtain

$$\sum_{l=0}^{+\infty} Z_{\gamma}(lh)\omega^l = -\sum_{j=1}^{nm} r_{0j}\sum_{s=j}^{+\infty} Z_{\gamma}\left((s-j)h\right)\omega^{s} - \sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}\sum_{s=j}^{+\infty} Z_{\gamma-i}\left((s-j)h\right)\omega^{s}.$$

By changing the order of summation, we have

$$\sum_{l=0}^{+\infty} Z_{\gamma}(lh)\omega^l = -\sum_{s=0}^{+\infty}\left(\sum_{j=1}^{\min\{s,nm\}} r_{0j}Z_{\gamma}\left((s-j)h\right) + \sum_{i=1}^{n}\sum_{j=0}^{\min\{s,nm\}} r_{ij}Z_{\gamma-i}\left((s-j)h\right)\right)\omega^{s}.$$

Comparing coefficients of the same power of ω yields

$$Z_{\gamma}(lh) = -\sum_{j=1}^{\theta_s} r_{0j}Z_{\gamma}\left((l-j)h\right) - \sum_{i=1}^{n}\sum_{j=0}^{\theta_s} r_{ij}Z_{\gamma-i}\left((l-j)h\right)$$

for $l=0,1,\ldots$; $\gamma=n+1,n+2,\ldots$; $\theta_s=\min\{s,nm\}$. This completes the proof of Lemma 2.

C. Proofs of Lemmas 3 and 4

We leave it to the reader to verify that the proofs of Lemmas 3 and 4 are similar to those of Lemmas 1 and 2.

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