RELATIVE CONTROLLABILITY OF STATIONARY HYBRID SYSTEMS*

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Abstract. The paper deals with the problem of relative controllability for hybrid linear stationary systems. For such systems, we introduce the determining equations and give solution representations into series their determining equation solutions. Then algebraic properties of the determining equation solutions are investigated, in particular, the well-known Hamilton-Cayley matrix theorem is extended to the solutions of the system determining equations. As a result a parametric criterion for the relative controllability is established.

Key Words. hybrid systems, determining equations, controllability.

1. INTRODUCTION

Many complex control systems (air traffic control, chemical engineering, transportation, manufacturing systems, robotics and others) are described both differential and algebraic equations with delay. They are examples of "hybrid" systems. Hybrid system under consideration consists of differential and different matrix equations, so we deal with both continuous and discrete variables. But it should be noted that the term "hybrid systems" has been widely used in the literature in various senses [1-6].

In the paper, we consider a linear stationary hybrid system of the form

$$\begin{aligned} \dot{x}(t) &= A_{11}x(t) + A_{12}y(t) + B_1u(t), \ t > 0, \\ y(t) &= A_{21}x(t) + A_{22}y(t-h) + B_2u(t), \ t \geq 0, \end{aligned} \quad (1)$$

here $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$, $y(t) \in \mathbb{R}^m$, $t \geq 0$; $A_{11} \in \mathbb{R}^{n\times n}$, $A_{12} \in \mathbb{R}^{n\times m}$, $A_{21} \in \mathbb{R}^{m\times n}$, $A_{22} \in \mathbb{R}^{m\times m}$, $B_1 \in \mathbb{R}^{n\times r}$, $B_2 \in \mathbb{R}^{m\times r}$ are constant (real) matrices and $u(t)$ is a piecewise continuous r-vector function (control), h is a constant delay , $h > 0$. We regard an absolute continuous n-vector function $x(\cdot)$ and a piecewise continuous m-vector function $y(\cdot)$ as a solution of System (1) if they satisfy the first equations (1) for almost everywhere $t > 0$ and the second one for $t \geq 0$.

System (1) should be completed with initial conditions

$$x(+0) = x_0 \in \mathbb{R}^n, \ y(\tau) = \psi(\tau), \ \tau \in [-h, 0) \quad (2)$$

where $\psi(\cdot)$ is a piecewise continuous m-vector function in the interval $[-h,0]$.

Let us introduce notation:

$T_t = \left[\dfrac{t}{h}\right]$, where symbol $[z]$ means entire part of the number z;

$\mathbb{R}^{k\times d}[\omega]$ is the set of k by d matrices over the ring of polynomials in ω;

I_n is the identity n by n matrix.

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Consider the determining equations of System (1):

$$X_k(t) = A_{11}X_{k-1}(t) + A_{12}Y_{k-1}(t) + B_1U_{k-1}(t) \qquad (3)$$

$$Y_k(t) = A_{21}X_k(t) + A_{22}Y_k(t-h) + B_2U_k(t) \qquad (4)$$

$k = 0,1,...; \quad t \ge 0$

with initial conditions

$X_k(t) = 0,\ Y_k(t) = 0\ \ if\ \ k < 0\ \ or\ \ t < 0$;

$U_0(0) = I_r,\ U_k(t) = 0\ \ if\ \ k^2 + t^2 \ne 0$.

One can prove that the solution $x(\cdot)$, $y(\cdot)$ of System (1) with initial conditions (2) can be represented in the following form [4]:

$$x(t) = \sum_{k=0}^{+\infty} \sum_{\substack{i \\ t-ik>0}} X_{k+1}(ih) \int_0^{t-ih} \frac{(t-\tau-ih)^k}{k!} u(\tau)d\tau +$$
$$+s_1(t, x_0, \psi),\ \ t > 0,$$

$$y(t) = \sum_{k=0}^{+\infty} \sum_{\substack{i \\ t-ik>0}} Y_{k+1}(ih) \int_0^{t-ih} \frac{(t-\tau-ih)^k}{k!} u(\tau)d\tau +$$
$$+ \sum_{\substack{i \\ t-ik\ge 0}} Y_0(ih)u(t-ih) + s_2(t, x_0, \psi),\ t \ge 0,$$

where vector functions $s_1(t, x_0, \psi)$ and $s_2(t, x_0, \psi)$ do not depend on control.

2. ALGEBRAIC PROPERTIES OF THE DETERMINING EQUATION SOLUTIONS

Denoting

$$A(\omega) = A_{11} + A_{12}\left(I_m - A_{22}\omega\right)^{-1} A_{21} \in \mathbb{R}^{n\times n}[\omega],$$
$$B(\omega) = B_1 + A_{12}\left(I_m - A_{22}\omega\right)^{-1} B_2 \in \mathbb{R}^{n\times r}[\omega],$$
$$C(\omega) = \left(I_m - A_{22}\omega\right)^{-1} A_{21} \in \mathbb{R}^{m\times n}[\omega],$$

we can state [4]:

Lemma 1. The following identities take place

$$\left(A(\omega)\right)^k B(\omega) \equiv \sum_{j=0}^{+\infty} X_{k+1}(jh)\omega^j,\ \ k = 0,1,... \qquad (5)$$

$$C(\omega)\left(A(\omega)\right)^k B(\omega) \equiv \sum_{j=0}^{+\infty} Y_{k+1}(jh)\omega^j,\ k = 0,1,... \qquad (6)$$

$$\left(I_m - A_{22}\omega\right)^{-1} B_2 \equiv \sum_{j=0}^{+\infty} Y_0(jh)\omega^j \qquad (7)$$

$|\omega| < \omega_1$, for some sufficiently positive number ω_1.

Consider the characteristic equation of $A(\omega)$:

$$0 = \Delta(\lambda) = \det\left(\lambda I_n - A_{11} - A_{12}\left(I_m - A_{22}\omega\right)^{-1} A_{21}\right) =$$
$$= \frac{1}{(\alpha(\omega))^n} \det\left(\lambda\alpha(\omega)I_n - \alpha(\omega)A_{11} - A_{12}Q_1^*(\omega)A_{21}\right).$$

Here $Q_1^*(\omega) \in \mathbb{R}^{m\times m}[\omega]$ is the adjoint of matrix $\left(I_m - A_{22}\omega\right)$, i.e. $\left(I_m - A_{22}\omega\right)Q_1^*(\omega) = \alpha(\omega)$,
where $\alpha(\omega) = \det\left(I_m - A_{22}\omega\right) \in \mathbb{R}^{1\times 1}[\omega]$.
Suppose $|\omega| < \omega_1$ for some sufficiently positive number, then

$$\sum_{i=0}^{n} \sum_{j=0}^{nm} r_{ij}\lambda^{n-i}\omega^j = 0 \qquad (8)$$

where r_{ij} $(i = 0,1,...,n;\ j = 0,1,...,nm)$ are real number with $r_{00} = 1$.
Equation (8) can be rewritten as

$$\lambda^n = -\sum_{j=1}^{nm} r_{0j}\lambda^n\omega^j - \sum_{i=1}^{n} \sum_{j=0}^{nm} r_{ij}\lambda^{n-i}\omega^j,\ |\omega| < \omega_1 \qquad (9)$$

Lemma 2. The solutions $X_\gamma(t),\ Y_\gamma(t),\ t \ge 0$, of the determining equations (3), (4) satisfy the characteristic equation (9), i.e.

$$X_\gamma(kh) = -\sum_{j=1}^{\theta_k} r_{0j}X_\gamma((k-j)h) - \sum_{i=1}^{n} \sum_{j=0}^{\theta_k} r_{ij}X_{\gamma-i}((k-j)h) \qquad (10)$$

$$Y_\gamma(kh) = -\sum_{j=1}^{\theta_k} r_{0j}Y_\gamma((k-j)h) - \sum_{i=1}^{n} \sum_{j=0}^{\theta_k} r_{ij}Y_{\gamma-i}((k-j)h) \qquad (11)$$

for $\gamma = n+1, n+2, ...$ and $k = 0,1,...$; $\theta_k = \min\{k, nm\}$.

Proof. By the Hamilton-Cayley matrix theorem, we obtain

$$(A(\omega))^n = -\sum_{j=1}^{nm} r_{0j}(A(\omega))^n \omega^j - \sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}(A(\omega))^{n-i}\omega^j \quad (12)$$

If we multiply (12) by matrix $\left(A(\omega)\right)^{\beta-1}B(\omega),\ \beta\in\mathbb{N}$, from the right, then

$$(A(\omega))^{n+\beta-1}B(\omega) = -\sum_{j=1}^{nm} r_{0j}(A(\omega))^{n+\beta-1}B(\omega)\omega^j -$$
$$-\sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}(A(\omega))^{n+\beta-i-1}B(\omega)\omega^j,\ |\omega| < \omega_1 .$$

By (5), it can be rewritten as

$$\sum_{k=0}^{+\infty} X_{n+\beta}(kh)\omega^k = -\sum_{j=1}^{nm} r_{0j}\sum_{k=0}^{+\infty} X_{n+\beta}(kh)\omega^k\omega^j -$$
$$-\sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}\sum_{k=0}^{+\infty} X_{n+\beta-i}(kh)\omega^k\omega^j$$

or, by denoting $n+\beta=\gamma$ and setting $k+j=s\ (k=s-j\geq 0)$, we get

$$\sum_{k=0}^{+\infty} X_{\gamma}(kh)\omega^k = -\sum_{j=1}^{nm} r_{0j}\sum_{s=j}^{+\infty} X_{\gamma}((s-j)h)\omega^s -$$
$$-\sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}\sum_{s=j}^{+\infty} X_{\gamma-i}((s-j)h)\omega^s$$

and, changing the order of summing, we have

$$\sum_{k=0}^{+\infty} X_{\gamma}(kh)\omega^k = -\sum_{s=0}^{+\infty}\left(\sum_{j=1}^{\min\{s,nm\}} r_{0j}X_{\gamma}((s-j)h) + \right.$$
$$\left. + \sum_{i=1}^{n}\sum_{j=0}^{\min\{s,nm\}} r_{ij}X_{\gamma-i}((s-j)h)\right)\omega^s .$$

If we equate the coefficients of the same powers of ω, then we obtain (10) being true.
So, (10) holds and we return to (11).
Multiplying both sides of (12) by matrix $\left(A(\omega)\right)^{\beta-1}B(\omega),\ \beta\in\mathbb{N}$, from the right and by matrix $C(\omega)$ from the left, we have

$$C(\omega)(A(\omega))^{n+\beta-1}B(\omega) = -\sum_{j=1}^{nm} r_{0j}C(\omega)(A(\omega))^{n+\beta-1}B(\omega)\omega^j -$$
$$-\sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}C(\omega)(A(\omega))^{n+\beta-i-1}B(\omega)\omega^j$$

and, using the reasons as above, we can prove identity (11).
This completes the proof of Lemma 2.

Introduce notation:

$$D(\omega) = \left(I_m - A_{21}\left(I_n - A_{11}\omega\right)^{-1}A_{12}\omega\right)^{-1}A_{22}, \quad (13)$$

$$F(\omega) = A_{21}\left(I_n - (A_{11}+A_{12}A_{21})\omega\right)^{-1}\left(A_{12}B_2 + B_1\right)\omega + B_2, \quad (14)$$

$$G(\omega) = \left(I_n - A_{11}\omega\right)^{-1}A_{12}\omega \quad (15)$$

Lemma 3. The following identities are valid:

$$\left(D(\omega)\right)^j F(\omega) \equiv \sum_{k=0}^{+\infty} Y_k(jh)\omega^k,\ j=0,1,... \quad (16)$$

$$G(\omega)\left(D(\omega)\right)^j F(\omega) \equiv \sum_{k=0}^{+\infty} X_k(jh)\omega^k,\ j=1,2,... \quad (17)$$

$$\left(I_n - (A_{11}+A_{12}A_{21})\omega\right)^{-1}\left(A_{12}B_2 + B_1\right)\omega \equiv \sum_{k=0}^{+\infty} X_k(0)\omega^k \quad (18)$$

$|\omega| < \omega_1$, where ω_1 is a sufficiently small positive number.

Proof. Multiplying each determining equations (3), (4) at moment $t=0$ by ω^k and summing over all k from 0 to $+\infty$, we obtain

$$\sum_{k=0}^{+\infty} X_k(0)\omega^k = \sum_{k=0}^{+\infty} A_{11}X_{k-1}(0)\omega^k +$$
$$+\sum_{k=0}^{+\infty} A_{12}Y_{k-1}(0)\omega^k + \sum_{k=0}^{+\infty} B_1U_{k-1}(0)\omega^k \quad (19)$$

$$\sum_{k=0}^{+\infty} Y_k(0)\omega^k = \sum_{k=0}^{+\infty} A_{21}X_k(0)\omega^k + \sum_{k=0}^{+\infty} B_2U_k(0)\omega^k \quad (20)$$

and, by changing $k-1=s$, the expression (19) can be rewritten as

$$\sum_{k=0}^{+\infty} X_k(0)\omega^k = \sum_{s=-1}^{+\infty} A_{11}X_s(0)\omega^{s+1} + \sum_{s=-1}^{+\infty} A_{12}Y_s(0)\omega^{s+1} + B_1\omega .$$

It follows from here and (20) that

$$\sum_{k=0}^{+\infty} X_k(0)\omega^k \equiv \left(I_n - (A_{11}+A_{12}A_{21})\omega\right)^{-1}\left(A_{12}B_2 + B_1\right)\omega$$

and identity (18) holds.
Let us prove identity (16) by induction.
Base step. Combining (18) and (20), we get

$$\sum_{k=0}^{+\infty} Y_k(0)\omega^k = A_{21}\left(I_n - (A_{11}+A_{12}A_{21})\omega\right)^{-1}\left(A_{12}B_2 + B_1\right)\omega + B_2 ,$$

so, (16) holds for $j=0$. Suppose it is true for $j=p-1$ and any natural number p, i.e.

$$\left(D(\omega)\right)^{p-1} F(\omega) \equiv \sum_{k=0}^{+\infty} Y_k((p-1)h)\omega^k \quad (21)$$

Let us now show that (16) is true for $j=p$.

Multiplying each equation of (3), (4) at $t = ph$ by ω^k and summing over all k from 0 to $+\infty$, we obtain

$$\sum_{k=0}^{+\infty} X_k(ph)\omega^k = \sum_{k=0}^{+\infty} A_{11}X_{k-1}(ph)\omega^k + \sum_{k=0}^{+\infty} A_{12}Y_{k-1}(ph)\omega^k \tag{22}$$

$$\sum_{k=0}^{+\infty} Y_k(ph)\omega^k = \sum_{k=0}^{+\infty} A_{21}X_k(ph)\omega^k + \sum_{k=0}^{+\infty} A_{22}Y_k(ph-h)\omega^k \tag{23}$$

By setting $s = k-1$, (22) can be rewritten as

$$\sum_{k=0}^{+\infty} X_k(ph)\omega^k = \sum_{s=-1}^{+\infty} A_{11}X_s(ph)\omega^{s+1} + \sum_{s=-1}^{+\infty} A_{12}Y_s(ph)\omega^{s+1}.$$

Therefore, we get

$$\sum_{k=0}^{+\infty} X_k(ph)\omega^k = \left(I_n - A_{11}\omega\right)^{-1} A_{12}\omega \sum_{k=0}^{+\infty} Y_k(ph)\omega^k \tag{24}$$

Combining (23) and (24), we obtain

$$\sum_{k=0}^{+\infty} Y_k(ph)\omega^k = A_{21}\left(I_n - A_{11}\omega\right)^{-1} A_{12}\omega \sum_{k=0}^{+\infty} Y_k(ph)\omega^k + $$
$$+\sum_{k=0}^{+\infty} A_{22}Y_k(ph-h)\omega^k$$

and, taking into account (13), we have

$$\sum_{k=0}^{+\infty} Y_k(ph)\omega^k = D(\omega)\sum_{k=0}^{+\infty} Y_k(ph-h)\omega^k$$

that, by (21), implies

$$\left(D(\omega)\right)^p F(\omega) \equiv \sum_{k=0}^{+\infty} Y_k(ph)\omega^k .$$

This proves (16) for $j = p$. By induction, (16) is true for $j = 0,1,\ldots$. Combining (15), (16), and (24), we conclude that (17) is also true. Hence, the proof is completed.

By definition, put

$$\beta(\omega) = \det\left(I_n - A_{11}\omega\right),$$

$$\mu(\omega) = \det\left(I_m\beta(\omega) - A_{21}Q_2^*(\omega)A_{12}\omega\right).$$

Let $Q_2^*(\omega) \in \mathbb{R}^{n\times n}[\omega]$ and $Q_3^*(\omega) \in \mathbb{R}^{m\times m}[\omega]$ be the adjoint matrices of matrices $\left(I_n - A_{11}\omega\right)$ and $\left(I_m\beta(\omega) - A_{21}Q_2^*(\omega)A_{12}\omega\right)$ respectively.

Consider the characteristic equation of $D(\omega)$:

$$0 = \Delta(\lambda) = \det\left(\lambda I_m - D(\omega)\right) =$$
$$= \det\left(\lambda I_m - \left(I_m - A_{21}\frac{Q_2^*(\omega)}{\beta(\omega)}A_{12}\omega\right)^{-1} A_{22}\right) =$$
$$= \det\left(\lambda I_m - \beta(\omega)\left(I_m\beta(\omega) - A_{21}Q_2^*(\omega)A_{12}\omega\right)^{-1} A_{22}\right)$$

That, for ω_1 being sufficiently small positive number, it is equivalent to

$$\sum_{i=0}^{m}\sum_{j=0}^{nm^2} p_{ij}\lambda^{m-i}\omega^j = 0,\ |\omega| < \omega_1,\ p_{00} = 1 \tag{25}$$

Rewrite (25) as

$$\lambda^m = -\sum_{j=1}^{nm^2} p_{0j}\lambda^m\omega^j - \sum_{i=1}^{m}\sum_{j=0}^{nm^2} p_{ij}\lambda^{m-i}\omega^j .$$

Lemma 4. The solutions of the determining equations (3), (4) satisfy the following conditions:

$$X_k((\nu+1)h) = -\sum_{j=1}^{\min\{k,nm^2\}} p_{0j}X_{k-j}((\nu+1)h) -$$
$$-\sum_{i=1}^{m}\sum_{j=0}^{\min\{k,nm^2\}} p_{ij}X_{k-j}((\nu+1-i)h),$$

$$Y_k(\nu h) = -\sum_{j=1}^{\min\{k,nm^2\}} p_{0j}Y_{k-j}(\nu h) -$$
$$-\sum_{i=1}^{m}\sum_{j=0}^{\min\{k,nm^2\}} p_{ij}Y_{k-j}((\nu-i)h),$$

for $k = 0,1,\ldots$ and $\nu = m, m+1,\ldots$.

The proof of Lemma 4 is very similar to Lemma 2 one and can be omitted.

Remark 1. Lemma 2 for $\gamma = n+1$ and Lemma 4 for $\nu = m$ can be regarded as generalizations of the well-known Hamilton-Cayley matrix theorem to solutions of the determining equations (3), (4).

3. PARAMETRIC CRITERION FOR H-t_1–CONTROLLABILITY OF STATIONARY HYBRID SYSTEMS

Let H be an arbitrary p by $(n+m)$ matrix.

Definition 1. System (1) is called H-t_1-controllable for $t_1 > 0$ if for any vector $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in R^{n+m}$ and for any initial conditions (2) there exists a piecewise continuous control $u(\cdot)$ such that the condition $H\begin{bmatrix} x(t_1) \\ y(t_1) \end{bmatrix} = H\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ holds for the corresponding solution $x(t),\ y(t)$ of the system.

We say that the system is

(i) relatively t_1 – controllable if $H = I_{n+m}$;
(ii) relatively t_1 – controllable in x if $H = [I_n \quad 0]$;
(iii) relatively t_1 – controllable in y if $H = [0 \quad I_m]$.

For simplicity, we put

$$x_0 = 0,\ \psi(\tau) \equiv 0 \ \text{ for } \tau \in [-h,0] \tag{26}$$

Then the corresponding solution of the system (1), (26) can be represented in the form

$$x(t) = \\ = \sum_{k=0}^{+\infty} \left(\sum_{j=0}^{T_t-1} \int_{t-(j+1)h}^{t-jh} \sum_{i=0}^{j} X_{k+1}(ih) \frac{(t-\tau-ih)^k}{k!} u(\tau)\, d\tau + \right. \\ \left. + \int_0^{t-T_t h} \sum_{i=0}^{T_t} X_{k+1}(ih) \frac{(t-\tau-ih)^k}{k!} u(\tau)\, d\tau \right),\quad t>0,$$

$$y(t) = \\ = \sum_{k=0}^{+\infty} \left(\sum_{j=0}^{T_t-1} \int_{t-(j+1)h}^{t-jh} \sum_{i=0}^{j} Y_{k+1}(ih) \frac{(t-\tau-ih)^k}{k!} u(\tau)\, d\tau + \right. \\ \left. + \int_0^{t-T_t h} \sum_{i=0}^{T_t} Y_{k+1}(ih) \frac{(t-\tau-ih)^k}{k!} u(\tau)\, d\tau \right) + \\ + \sum_{i=0}^{T_t} Y_0(ih)\, u(t-ih),\ t \ge 0 .$$

The H –attainability set $K(t_1)$ of System (1), (26) at the moment t_1 is described as follows

$$K(t_1) = \\ = \left\{ \begin{aligned} & \gamma \in \mathbb{R}^p : \gamma = \\ & = H \sum_{k=0}^{+\infty} \sum_{j=0}^{T_{t_1}-1} \int_{t_1-(j+1)h}^{t_1-jh} \sum_{i=0}^{j} \begin{bmatrix} X_{k+1}(ih) \\ Y_{k+1}(ih) \end{bmatrix} \frac{(t_1-\tau-ih)^k}{k!} u(\tau) d\tau + \\ & + H \sum_{k=0}^{+\infty} \int_0^{t_1-T_{t_1}h} \sum_{i=0}^{T_{t_1}} \begin{bmatrix} X_{k+1}(ih) \\ Y_{k+1}(ih) \end{bmatrix} \frac{(t_1-\tau-ih)^k}{k!} u(\tau)\, d\tau + \\ & + H \sum_{j=0}^{T_{t_1}} \begin{bmatrix} 0 \\ Y_0(jh) \end{bmatrix} u(t_1 - jh),\ \forall u(\cdot) \in U(\cdot) \end{aligned} \right\}.$$

Here $U(\cdot)$ is the set of piecewise continuous r-vector-function in the interval $[0,\, t_1]$ and $K_0 = \{H\mu : \forall \mu \in R^{m+n}\}$ is the linear span of the columns of matrix H. Then H-t_1-controllability of the system is equivalent to the inclusion $K(t_1) \supset K_0$ or $(K(t_1))^{\perp} \subset K_0^{\perp}$ for orthogonal complements. Then we state the following

Theorem 1. System (1) is H-t_1-controllable if and only if for every vector $g \in \mathbb{R}^p$ such that

$$g'H \sum_{k=0}^{+\infty} \sum_{i=0}^{j} \frac{(t_1-\tau-ih)^k}{k!} \begin{bmatrix} X_{k+1}(ih) \\ Y_{k+1}(ih) \end{bmatrix} = 0,\ \tau \in (t_1-(j+1)h, t_1-jh),\ \tau > 0,$$

$$g'H \begin{bmatrix} 0 \\ Y_0(jh) \end{bmatrix} = 0,\ \ j = 0,\dots,T_{t_1},$$

the condition $g'H = 0$ also takes place.

Proof. We have:

$$\left(\forall g \in \mathbb{R}^p,\ g'K(t_1) = 0, \Rightarrow g'K_0 = 0 \right) \Leftrightarrow$$

$$\Leftrightarrow \left(\forall g \in \mathbb{R}^p, \right. \\ g'H \sum_{k=0}^{+\infty} \sum_{j=0}^{T_{t_1}-1} \int_{t_1-(j+1)h}^{t_1-jh} \sum_{i=0}^{j} \begin{bmatrix} X_{k+1}(ih) \\ Y_{k+1}(ih) \end{bmatrix} \frac{(t_1-\tau-ih)^k}{k!} u(\tau) d\tau + \\ + g'H \sum_{k=0}^{+\infty} \int_0^{t_1-T_{t_1}h} \sum_{i=0}^{T_{t_1}} \begin{bmatrix} X_{k+1}(ih) \\ Y_{k+1}(ih) \end{bmatrix} \frac{(t_1-\tau-ih)^k}{k!} u(\tau) d\tau + \\ \left. + g'H \sum_{j=0}^{T_{t_1}} \begin{bmatrix} 0 \\ Y_0(jh) \end{bmatrix} u(t_1-jh) = 0,\ \forall u(\cdot) \in U(\cdot), \Rightarrow g'H = 0 \right).$$

Setting

$$u(\tau) = \left(g'H \sum_{k=0}^{+\infty} \sum_{i=0}^{j} \frac{(t_1-\tau-ih)^k}{k!} \begin{bmatrix} X_{k+1}(ih) \\ Y_{k+1}(ih) \end{bmatrix} \right)',$$

$$\tau \in (t_1-(j+1)h,\, t_1-jh),\ \tau > 0,\ j = 0,\dots,T_{t_1},$$

$$u(t_1 - jh) = \left(g'H \begin{bmatrix} 0 \\ Y_0(jh) \end{bmatrix} \right)',\ \ j = 0,\dots,T_{t_1},$$

we obtain

$$\left(\forall g \in \mathbb{R}^p, \right. \\ \sum_{j=0}^{T_{t_1}-1} \int_{t_1-(j+1)h}^{t_1-jh} \left\| g'H \sum_{k=0}^{+\infty} \sum_{i=0}^{j} \frac{(t_1-\tau-ih)^k}{k!} \begin{bmatrix} X_{k+1}(ih) \\ Y_{k+1}(ih) \end{bmatrix} \right\|^2 d\tau + \\ + \int_0^{t_1-T_{t_1}h} \left\| g'H \sum_{k=0}^{+\infty} \sum_{i=0}^{T_{t_1}} \frac{(t_1-\tau-ih)^k}{k!} \begin{bmatrix} X_{k+1}(ih) \\ Y_{k+1}(ih) \end{bmatrix} \right\|^2 d\tau + \\ \left. + \sum_{j=0}^{T_{t_1}} \left\| g'H \begin{bmatrix} 0 \\ Y_0(jh) \end{bmatrix} \right\|^2 = 0, \Rightarrow g'H = 0 \right).$$

This finishes the proof.

Remark 2. The statement of Theorem 1 gives an implicit criterion for H-t_1-controllability of stationary hybrid systems.

For the sequel, we need the following result.

Lemma 5. The conditions

$$g'H\sum_{k=0}^{+\infty}\sum_{i=0}^{j}\frac{(t-\tau-ih)^k}{k!}\begin{bmatrix}X_{k+1}(ih)\\Y_{k+1}(ih)\end{bmatrix}=0, \qquad (27)$$

$\tau\in(t-(j+1)h, t-jh), \quad \tau>0, \; j=0,1,...$

hold if and only if the conditions

$$g'H\begin{bmatrix}X_{k+1}(ih)\\Y_{k+1}(ih)\end{bmatrix}=0 \qquad \begin{matrix}k=0,1,...\\ i=0,1,...j\end{matrix} \quad \text{are valid.}$$

Proof. Sufficiency is evident. The proof of necessity is by induction over j. It is not difficult to see that the statement of Lemma 5 is true for $j=0$. Assume now that statement of Lemma 5 is true for $j=\alpha-1$, where $\alpha\in\mathbb{N}$. By the induction hypothesis, for $j=\alpha$, $\tau\in(t-(\alpha+1)h, t-\alpha h)$, the identity (27) takes the form

$$g'H\sum_{k=0}^{+\infty}\frac{(t-\tau)^k}{k!}\begin{bmatrix}X_{k+1}(0)\\Y_{k+1}(0)\end{bmatrix}+...+g'H\sum_{k=0}^{+\infty}\frac{(t-\tau-\alpha h)^k}{k!}\begin{bmatrix}X_{k+1}(\alpha h)\\Y_{k+1}(\alpha h)\end{bmatrix}=0.$$

By the inductive assumption, we obtain

$$g'H\sum_{k=0}^{+\infty}\frac{(t-\tau-\alpha h)^k}{k!}\begin{bmatrix}X_{k+1}(\alpha h)\\Y_{k+1}(\alpha h)\end{bmatrix}=0.$$

It follows from here, by differentiating μ times ($\mu=0,1,...$) with respect to τ, that

$$g'H\begin{bmatrix}X_{\mu+1}(\alpha h)\\Y_{\mu+1}(\alpha h)\end{bmatrix}+g'H\sum_{k=\mu+1}^{+\infty}\frac{(t-\tau-\alpha h)^{k-\mu}}{(k-\mu)!}\begin{bmatrix}X_{k+1}(\alpha h)\\Y_{k+1}(\alpha h)\end{bmatrix}=0$$

and we have as $\tau\to t-\alpha h-0$

$$g'H\begin{bmatrix}X_{\mu+1}(\alpha h)\\Y_{\mu+1}(\alpha h)\end{bmatrix}=0, \; \mu=0,1,... .$$

This completes the proof.

We can now formulate a parametric H-t_1-controllability criterion expressed in terms of the determining equation solutions.

Theorem 2. System (1) is H-t_1-controllable iff

$$rank\left[H\begin{bmatrix}X_k(ih)\\Y_k(ih)\end{bmatrix}, k=0,1,...,n;\; i=0,1,...,\min\{T_{t_1},m\};\, H\right]=$$
$$=rank\left[H\begin{bmatrix}X_k(ih)\\Y_k(ih)\end{bmatrix}, k=0,1,...,n;\; i=0,1...\min\{T_{t_1},m\}\right].$$

Corollary 1. System (1) is relatively t_1-controllable if and only if

$$rank\left[\begin{bmatrix}X_k(ih)\\Y_k(ih)\end{bmatrix}, k=0,1,...,n;\; i=0,1,...,\min\{T_{t_1},m\}\right]=(m+n)\cdot$$

Corollary 2. System (1) is relatively t_1-controllable in x if and only if

$$rank\left[X_k(ih),\, k=1,2,...,n;\; i=0,1,...,\min\{T_{t_1},m\}\right]=n\,.$$

Corollary 3. System (1) is relatively t_1-controllable in y if and only if

$$rank\left[Y_k(ih),\, k=0,1,...,n;\; i=0,1,...,\min\{T_{t_1},m\}\right]=m\,.$$

4. CONCLUDING REMARKS

In the paper, we have investigated algebraic properties of determining equations for the simplest hybrid systems. The results considered have been applied to obtaining parametric criteria for relative controllability of such a system. Similarly, one can study a dual observability problem.

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