# HYBRID CONTROL AND OBSERVATION SYSTEMS IN SYMMETRIC FORM

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*Abstract: The paper considers several models of linear hybrid systems described by discrete-difference and difference-differential equations with control. Special attention is paid to the difference-differential hybrid systems in symmetric form. For solutions of such systems, a variation-of-constants formula is proposed and the relative controllability-observability principle is established. For stationary systems, we introduce the determining equations and present solutions in the form of series of their determining equation solutions. Then algebraic properties of solutions of determining equation are investigated, in particular, the well-known Hamilton-Cayley matrix theorem is extended to the solutions of the system of determining equations. As a result, parametric criteria for the relative controllability and relative observability are given.*

## 1 Introduction

Many complex control systems (air traffic control, chemical engineering, transportation, manufacturing systems, robotics, and others) are described both in differential and algebraic equations with delay. They are examples of “hybrid” systems. It should be noted that the term “hybrid systems” has been widely used in the literature in various senses [1-4]. Generally speaking, “hybridness” reflects dual structure of the systems. At present, hybrid systems are studied separately, that is continuous analysis methods are used to solve continuous systems and discrete analysis methods are employed for discrete systems. The aim of the paper is to give a unified approach to the construction of mathematical models of dynamical systems described by discrete, difference and differential equations, taking difference-differential hybrid systems as standard systems. Consider, for example, the simplest discrete-continuous hybrid system

$$\begin{aligned} &\dot{x}(t) = A_{11}x(t) + A_{12}y[k],\ t \in [kh,(k+1)h], \\ &y[k] = A_{21}x(kh) + A_{22}y[k-1],\ k = 0,1,... \end{aligned} \qquad (1)$$

with initial conditions of the form

$$x(+0) = x(0) = x_0, \quad y[-1] = y_0,$$

where $x(t) \in \mathbb{R}^n$ , $u(t) \in \mathbb{R}^r$, $y(t) \in \mathbb{R}^m$,

and $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$ are constant matrices. Denoting

$$\tilde{y}(t) = \begin{bmatrix} x(kh) \\ y[k] \end{bmatrix} \text{ for } t \in [kh,(k+1)h),\ k = 0,1,...,$$

where

$$x(kh) = e^{A_{11}(kh-(k-1)h)}x(kh-h) + \int_{kh-h}^{kh} e^{A_{11}(kh-\tau)}A_{12}y[k-1]d\tau,$$

and taking initial conditions as follows

$$x(0) = x_0,$$

$$\tilde{y}(\tau) = \begin{bmatrix} e^{-A_{11}h}\left(x_0 - \int_0^h e^{A_{11}(h-\tau)}A_{12}y_0 d\tau\right) \\ y_0 \end{bmatrix},\ \tau \in [-h,0),$$

it is not difficult to see that if $y(t) = \tilde{y}(t),\ t \geq 0$. System (1) can be represented as the following difference-differential hybrid system in normal form [4]

$$\begin{aligned} &\dot{x}(t) = A_{11}(t)x(t) + A_{12}(t)y(t) + B_1(t)u(t), \\ &y(t) = A_{21}(t)x(t) + A_{22}(t)y(t-h) + B_2(t)u(t),\ t \geq t_0, \end{aligned}$$

with initial conditions

$$x(t_0+0) = x(t_0) = x_0,\ y(\tau) = \psi(\tau),\ \tau \in [t_0-h,t_0).$$

That is why we pay a special attention to hybrid systems, consisting of differential and difference equations. Taking into account 2-D system reasons, we consider such a system in symmetric form with respect to the differential and shift operators:

$$\begin{aligned}&\dot{x}(t) = A_{11}(t)x(t) + A_{12}(t)y(t) + B_1(t)u(t),\\ &y(t+h) = A_{21}(t)x(t) + A_{22}(t)y(t) + B_2(t)u(t), \quad t \ge t_0,\end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$, $y(t+h) \in \mathbb{R}^m$, $t \ge t_0$; $h$ is a given positive number; components of the matrix-valued functions

$$A_{11}(t) \in \mathbb{R}^{n\times n}, A_{12}(t) \in \mathbb{R}^{n\times m}, A_{21}(t) \in \mathbb{R}^{m\times n},$$
$$A_{22}(t) \in \mathbb{R}^{m\times m},\ B_1(t) \in \mathbb{R}^{n\times r},\ B_2(t) \in \mathbb{R}^{m\times r}$$

are piecewise continuous in each compact interval $[a,b] \in [t_0, +\infty)$.

For System (2) we consider the following initial-valued problem as

$$x(t_0+0) = x_0, y(\tau) = \psi(\tau),\ \tau \in [t_0, t_0+h), \tag{3}$$

where $x_0 \in \mathbb{R}^n$, $\psi$ is a piecewise continuous $m-$ vector function in $[t_0, t_0+h]$.

A solution $x(t; t_0, x_0, \psi, u)$, $y(t+h; t_0, x_0, \psi, u)$ for $t \ge t_0$ is defined as follows: $x(\cdot)$ is continuous $n$-vector function and $y(\cdot)$ is piecewise continuous $m$-vector function, satisfying the first equation of (1) for $t \ge t_0$ except at the points $t = t_0 + kh,\ k = 0,1,...$ and the second equation of (1) for $t \ge t_0$.

For differentiable $\psi$ and $u$, System (2) can be reduced to a neutral type time-delay equation and, as a result, the variation-of-constants formula can be used to represent the system solutions. In general, the situation is more complicated. However, by using the method of steps, one can state that the solution of System (2) corresponding to initial problem (3) and to the piecewise continuous control $u$ exists and is unique.

## 2 A variation–of–constants formula for hybrid systems

Let us denote $T_t = \left[\frac{t-t_0}{h}\right]$ and define matrix functions $X^*(t,\tau), Z^*(t,\tau), Y^*(t,\tau)$ as solutions of an adjoint system of the form:

$$-\frac{\partial X^*(t,\tau)}{\partial \tau} = X^*(t,\tau)A_{11}(\tau) + Y^*(t,\tau)A_{21}(\tau),$$
$$\tau \le t,\ \tau \ne t - kh,\ k = 1,2,...,T_t;$$
$$Y^*(t,\tau-h) = X^*(t,\tau)A_{12}(\tau) + Y^*(t,\tau)A_{22}(\tau), \quad \tau \le t;$$
$$Y^*(t,\tau) = 0, \quad \tau > t-h;$$
$$X^*(t, t-kh-0) - X^*(t, t-kh+0) =$$
$$= Z^*(t, t-kh)A_{21}(t-kh), k = 1,2,...,T_t;$$
$$Z^*(t, t-kh-h) = Z^*(t, t-kh)A_{22}(t-kh),\ k = 1,2,...,T_t - 1.$$

According to the adjoint system, solutions $X^*(t,\tau)$ and $Y^*(t,\tau)$ are piecewise-continuous with respect to argument $\tau$ matrix functions with jumps at points $\tau = t - kh,\ k = 1,2,...,T_t$. The matrix function $Z^*(t, t-kh)$ is regarded as discrete with respect to the second argument. The last three equations establish the initial and boundary conditions for the adjoint system under consideration.

*Theorem 1.* The solution of the system (2) with initial conditions (3) and piecewise continuous control $u(t), t > t_0$, exists, is unique and can be written as follows

$$X^*(t, t_0-0)x_0 + \int_{t_0}^{t_0+h} Y^*(t, \tau-h)\psi(\tau)d\tau +$$
$$+\int_{t_0}^{t} \left(X^*(t,\tau)B_1(\tau) + Y^*(t,\tau)B_2(\tau)\right)u(\tau)d\tau +$$
$$+\sum_{k=1}^{T_t} Z^*(t, t-kh)B_2(t-kh)u(t-kh) +$$
$$+Z^*(t, t-T_t h)\psi(t-T_t h) =$$
$$= \left\{\begin{array}{l} x(t) \ \ for\ t \ge t_0 \quad if\ \ X^*(t,t-0) = I_n \in \mathbb{R}^{n\times n} \\ \qquad\qquad and\ \ Z^*(t,t-h) = 0 \in \mathbb{R}^{n\times m}; \\ y(t)\ \ for\ t \ge t_0 + h\ \ if\ \ X^*(t,t-0) = 0 \in \mathbb{R}^{m\times n} \\ \qquad\qquad and\ \ Z^*(t,t-h) = I_m \in \mathbb{R}^{m\times m} \end{array}\right\}.$$

Further in the paper, we consider a linear stationary hybrid system, i.e.

$$A_{11}(t) = A_{11}, A_{12}(t) = A_{12}, A_{21}(t) = A_{21}, A_{22}(t) = A_{22},$$
$$B_1(t) = B_1, B_2(t) = B_2,\ t_0 = 0. \tag{4}$$

For stationary systems Theorem 1 can be detailed if we take into account that

$$X^*(t,\tau) = X(t-\tau),\ \ Y^*(t,\tau) = Y(t-\tau),\ \ Z^*(t,\tau) = Z(t-\tau)$$

and the adjoint system can be rewritten as

$$\dot{X}(t) = X(t)A_{11} + Y(t)A_{21},\ t \ge 0,\ t \ne kh,\ k = 1,...,T_t; \tag{5}$$
$$Y(t+h) = X(t)A_{12} + Y(t)A_{22}, \quad t \ge 0; \tag{6}$$
$$Y(t) = 0, \quad t < h; \tag{7}$$

$$X(kh)-X(kh-0)=Z(kh)A_{21}, k=1,2,\dots,T_t; \quad (8)$$
$$Z(kh+h)=Z(kh)A_{22}, \ \ k=1,2,\dots,T_t-1. \quad (9)$$

*Remark 1.* If $T_t=0$, then equations (8) and (9) disappear. Equation (9) disappears also when $T_t=1$. In (5) $X(t)$ have right hand derivative at the points $t=kh,\ k=1,2,\dots,T_t$.

*Theorem 2.* The solution of the system (2)–(4) with piecewise continuous control $u(t),\, t>0,$ exists, is unique and can be written as follows

$$x(t;\,x_0,\psi,u)=X(t)x_0+\int_0^h Y(t-\tau+h)\psi(\tau)d\tau+$$
$$+\int_0^t \big(X(t-\tau)B_1+Y(t-\tau)B_2\big)u(\tau)d\tau \quad (10)$$
$$for\ t\ge 0 \quad if\ \ X(0)=I_n\in\mathbb{R}^{n\times n},\ Z(h)=0\in\mathbb{R}^{n\times m};$$
$$y(t;\,x_0,\psi,u)=X(t)x_0+\int_0^h Y(t-\tau+h)\psi(\tau)d\tau+$$
$$+\int_0^t \big(X(t-\tau)B_1+Y(t-\tau)B_2\big)u(\tau)d\tau+$$
$$+Z(T_th)\psi(t-T_th)+\sum_{k=1}^{T_t} Z(kh)B_2u(t-kh) \quad (11)$$
$$for\ t\ge h \quad if\ \ X(0)=0\in\mathbb{R}^{m\times n},\ Z(h)=I_m\in\mathbb{R}^{m\times m}.$$

## 3 Algebraic properties of the determining equation solutions

Introduce determining equations of System (2)–(4):

$$\begin{aligned}X_{k+1}(t)&=A_{11}X_k(t)+A_{12}Y_k(t)+B_1U_k(t),\\ Y_k(t+h)&=A_{21}X_k(t)+A_{22}Y_k(t)+B_2U_k(t)\end{aligned} \quad (12)$$
$$k=-1,0,1,\dots; \quad t\ge -h,$$

with initial conditions

$$X_k(t)=0,\ Y_k(t)=0\ \ if\ \ k<0\ \ or\ \ t<0;$$
$$U_0(0)=I_r,\ U_k(t)=0\ \ if\ \ k^2+\ t^2\ne 0.$$

By induction, we can prove Lemmas 1 and 2.

*Lemma 1.* The following identities hold:

$$\left(A_{11}+A_{12}\omega\left(I_m-A_{22}\,\omega\right)^{-1}A_{21}\right)^{k-1}\times$$
$$\times\left(B_1+A_{12}\,\omega\left(I_m-A_{22}\,\omega\right)^{-1}B_2\right)\equiv\sum_{j=0}^{+\infty}X_k(jh)\omega^j;$$

$$\left(I_m-A_{22}\,\omega\right)^{-1}A_{21}\omega\left(A_{11}+A_{12}\omega\left(I_m-A_{22}\,\omega\right)^{-1}A_{21}\right)^{k-1}\times$$
$$\times\left(B_1+A_{12}\omega\left(I_m-A_{22}\,\omega\right)^{-1}B_2\right)\equiv\sum_{j=0}^{+\infty}Y_k(jh)\omega^j;$$
$$k=1,2,\dots$$
$$\left(I_m-A_{22}\,\omega\right)^{-1}B_2\omega\equiv\sum_{j=0}^{+\infty}Y_0(jh)\omega^j;$$

$$\left(A_{22}+A_{21}\omega\left(I_n-A_{11}\omega\right)^{-1}A_{12}\right)^{j-1}\times$$
$$\times\left(A_{21}\omega\left(I_n-A_{11}\omega\right)^{-1}B_1+B_2\right)\equiv\sum_{k=0}^{+\infty}Y_k(jh)\omega^k;$$
$$\left(I_n-A_{11}\omega\right)^{-1}A_{12}\omega\left(A_{22}+A_{21}\omega\left(I_n-A_{11}\omega\right)^{-1}A_{12}\right)^{j-1}\times$$
$$\times\left(A_{21}\omega\left(I_n-A_{11}\omega\right)^{-1}B_1+B_2\right)\equiv\sum_{k=0}^{+\infty}X_k(jh)\ \ \omega^k;$$
$$j=1,2,\dots;$$
$$\left(I_n-A_{11}\omega\right)^{-1}B_1\omega\equiv\sum_{k=0}^{+\infty}X_k(0)\ \ \omega^k;$$

$|\omega|<\omega_1$, $\omega_1$ is a sufficiently small positive number.

Denote:

$$\alpha(\omega)=\det\left(I_m-A_{22}\omega\right)\in\mathbb{R}^1[\omega],$$
$$\beta(\omega)=\det\left(I_n-A_{11}\omega\right)\in\mathbb{R}^1[\omega].$$

$Q_1^*(\omega)\in\mathbb{R}^{m\times m}[\omega]$ and $Q_2^*(\omega)\in\mathbb{R}^{n\times n}[\omega]$ are the adjoint of matrices $\left(I_m-A_{22}\omega\right)$ and $\left(I_n-A_{11}\omega\right)$ respectively.

Introduce the characteristic equation of the matrix $\left(A_{11}+A_{12}\omega\left(I_m-A_{22}\omega\right)^{-1}A_{21}\right)$:

$$0=\Delta(\lambda)=\det\left(\lambda I_n-A_{11}+A_{12}\omega\left(I_m-A_{22}\omega\right)^{-1}A_{21}\right)=$$
$$=\det\left(\lambda I_n-A_{11}-\frac{A_{12}\omega Q_1^*(\omega)A_{21}}{\alpha(\omega)}\right)=$$
$$=\frac{1}{\left(\alpha(\omega)\right)^n}\det\left(\lambda\alpha(\omega)I_n-A_{11}\alpha(\omega)-A_{12}\omega Q_1^*(\omega)A_{21}\right). \quad (13)$$

Consider the characteristic equation of the matrix $\left(A_{22}+A_{21}\omega\left(I_n-A_{11}\omega\right)^{-1}A_{12}\right)$:

$$0=\Delta(\lambda)=\det\left(\lambda I_m-A_{22}-A_{21}\omega\left(I_n-A_{11}\omega\right)^{-1}A_{12}\right)=$$

$$= \det\left(\lambda I_m - A_{22} - \frac{A_{21}\omega Q_2^*(\omega)A_{12}}{\beta(\omega)}\right) =$$

$$= \frac{1}{(\beta(\omega))^m}\det\left(\lambda\beta(\omega)I_m - A_{22}\beta(\omega) - A_{21}\omega Q_2^*(\omega)A_{12}\right). \quad (14)$$

Suppose $|\omega| < \omega_1$ where $\omega_1$ is a sufficiently small positive number. Then (13) and (14) can be represented as

$$\sum_{i=0}^{n}\sum_{j=0}^{nm} r_{ij}\lambda^{n-i}\omega^j = 0 \quad \text{and} \quad \sum_{i=0}^{m}\sum_{j=0}^{nm} p_{ij}\lambda^{m-i}\omega^j = 0$$

respectively.

Here $r_{ij}$ and $p_{kj}$, $i = 0,1,...,n;\ k = 0,1,...,m;\ j = 0,1,...,nm$, are real numbers with $r_{00} = 1$ and $p_{00} = 1$.

The last two equations can be rewritten as

$$\lambda^n = -\sum_{j=1}^{nm} r_{0j}\lambda^n\omega^j - \sum_{i=1}^{n}\sum_{j=0}^{nm} r_{ij}\lambda^{n-i}\omega^j,$$

$$\lambda^m = -\sum_{j=1}^{nm} p_{0j}\lambda^m\omega^j - \sum_{i=1}^{m}\sum_{j=0}^{nm} p_{ij}\lambda^{m-i}\omega^j,\ |\omega| < \omega_1.$$

*Lemma 2*. Solutions of the determining equations (12) satisfy the following conditions:

$$X_\gamma(kh) = -\sum_{j=1}^{\min\{k,\,nm\}} r_{0j}X_\gamma((k-j)h) -$$

$$-\sum_{i=1}^{n}\sum_{j=0}^{\min\{k,\,nm\}} r_{ij}X_{\gamma-i}((k-j)h),$$

$$Y_\gamma(kh) = -\sum_{j=1}^{\min\{k,\,nm\}} r_{0j}Y_\gamma((k-j)h) -$$

$$-\sum_{i=1}^{n}\sum_{j=0}^{\min\{k,\,nm\}} r_{ij}Y_{\gamma-i}((k-j)h),$$

$$X_k(\nu h) = -\sum_{j=1}^{\min\{k,\,nm\}} p_{0j}X_{k-j}(\nu h) -$$

$$-\sum_{i=1}^{m}\sum_{j=0}^{\min\{k,\,nm\}} p_{ij}X_{k-j}((\nu-i)h),$$

$$Y_k(\nu h) = -\sum_{j=1}^{\min\{k,\,nm\}} p_{0j}Y_{k-j}(\nu h) -$$

$$-\sum_{i=1}^{m}\sum_{j=0}^{\min\{k,\,nm\}} p_{ij}Y_{k-j}((\nu-i)h),$$

for $k = 0,1,...,$ $\nu = m+1,...$ and $\gamma = n+1,...$,

where $\sum_{k=i}^{j}(...) = 0$ if $j < i$.

For $\gamma = n+1$ and $\nu = m+1$ Lemma 2 can be regarded as a generalization of the well-known theorem of Hamilton-Cayley from matrix theory to solutions of the determining equations (12).

## 4 Solution increase estimate

*Theorem 3.* Suppose $\max_{t\in[0,h]}\|\psi(t)\| = M_1$ and $\|u(t)\| \le M_2 e^{\sigma t}$, $t \ge 0$ (where $M_1$, $M_2$ and $\sigma$ are positive constants), then there exist positive numbers $N$ and $\alpha$ such that all solutions of the system (2)–(4) satisfy the following conditions

$$\|x(t)\| \le Ne^{\alpha t}, \quad \|y(t)\| \le Ne^{\alpha t}, \quad t \ge 0,$$

where $N, \alpha$ are defined only by $M_1, M_2, \sigma$ and system parameters.

## 5 Solution representations into series of their determining equation solutions

*Theorem 4.* The solution of the system (2)–(4) with piecewise continuous control $u(\tau)$, $\tau \ge 0$, exists, is unique and can be represented by the following formulas:

$$x(t;x_0,\psi,u) = \sum_{k=0}^{+\infty}\sum_{\substack{i\\ t-ih\ge 0}} X_{k+1}(ih)\int_0^{t-ih}\frac{(t-\tau-ih)^k}{k!}u(\tau)d\tau +$$

$$+x(t;x_0,\psi,0), \quad t > 0, \quad (15)$$

$$y(t;x_0,\psi,u) = \sum_{k=0}^{+\infty}\sum_{\substack{i\\ t-ih\ge 0}} Y_{k+1}(ih)\int_0^{t-ih}\frac{(t-\tau-ih)^k}{k!}u(\tau)d\tau +$$

$$+\sum_{\substack{i\\ t-ih\ge 0}} Y_0(ih)u(t-ih) + y(t;x_0,\psi,0),\ t \ge 0, \quad (16)$$

where vector functions $x(t;x_0,\psi,0)$, $y(t;x_0,\psi,0)$ depend on initial data only.

*Proof.* From (9) and (12) we obtain

$$Y_0(jh) = (A_{22})^{j-1}B_2, \ j = 1,2,...$$

$$Z(jh) = Z(h)(A_{22})^{j-1}, \ j = 1,2,... .$$

Thus the last term in (11) can be revised

$$\sum_{j=1}^{T_t} Z(jh)B_2u(t-jh) = \sum_{j=1}^{T_t} Y_0(jh)u(t-jh)\,.$$

According to Theorem 3, we can apply Laplas transformation to the system (5)–(9). As a result, the following relations take place

$$X(0)+\sum_{k=1}^{+\infty}\left(X(kh)-X(kh-0)\right)e^{-pkh} =$$
$$= p\breve{X}(p)-\breve{X}(p)A_{11}-\breve{Y}(p)A_{21},$$
$$e^{ph}\breve{Y}(p)=\breve{X}(p)A_{12}+\breve{Y}(p)A_{22},$$

where $\operatorname{Re} p > \alpha$. Or

$$\breve{Y}(p)=A_{12}\breve{X}(p)\left(I_m e^{ph}-A_{22}\right)^{-1},$$
$$X(0)+\sum_{k=1}^{+\infty}\left(X(kh)-X(kh-0)\right)e^{-pkh} =$$
$$=\breve{X}(p)\left(pI_n-A_{11}-A_{12}e^{-ph}\left(I_m-A_{22}e^{-ph}\right)^{-1}A_{21}\right).$$

Last two relations can be rewritten as

$$\breve{X}(p)=\left(X(0)+\sum_{k=1}^{+\infty}\left(X(kh)-X(kh-0)\right)e^{-pkh}\right)\times$$
$$\times\frac{1}{p}\left(I_n-\frac{1}{p}A_{11}-\frac{1}{p}A_{12}e^{-ph}\left(I_m-A_{22}e^{-ph}\right)^{-1}A_{21}\right)^{-1}, \quad (22)$$

$$\breve{Y}(p)=\left(X(0)+\sum_{k=1}^{+\infty}\left(X(kh)-X(kh-0)\right)e^{-pkh}\right)\times$$
$$\times A_{12}e^{-ph}\left(I_m-A_{22}e^{-ph}\right)^{-1}\times$$
$$\times\frac{1}{p}\left(I_n-\frac{1}{p}A_{11}-\frac{1}{p}A_{12}\left(I_m-A_{22}e^{-ph}\right)^{-1}e^{-ph}A_{21}\right)^{-1}. \quad (23)$$

If we apply Laplas transformation to the term $\int_0^t\left(X(t-\tau)B_1+Y(t-\tau)B_2\right)u(\tau)d\tau$ of formulas (10), (11) and take into account (22), (23), then we have:

$$\left(\breve{X}(p)B_1+\breve{Y}(p)B_2\right)\breve{u}(p)=$$
$$=\left(X(0)+\sum_{k=1}^{+\infty}\left(X(kh)-X(kh-0)\right)e^{-pkh}\right)\times$$
$$\times\frac{1}{p}\left(I_n-\frac{1}{p}A_{11}-\frac{1}{p}A_{12}e^{-ph}\left(I_m-A_{22}e^{-ph}\right)^{-1}A_{21}\right)^{-1}\times$$
$$\times\left(B_1+A_{12}e^{-ph}\left(I_m-A_{22}e^{-ph}\right)^{-1}B_2\right)\breve{u}(p)=$$
$$=\left(X(0)+\sum_{k=1}^{+\infty}\left(X(kh)-X(kh-0)\right)e^{-pkh}\right)\times$$
$$\times\sum_{k=0}^{+\infty}\frac{1}{p^{k+1}}\left(A_{11}+A_{12}e^{-ph}\left(I_m-A_{22}e^{-ph}\right)^{-1}A_{21}\right)^k\times$$
$$\times\left(B_1+A_{12}e^{-ph}\left(I_m-A_{22}e^{-ph}\right)^{-1}B_2\right)\breve{u}(p)\,. \quad (24)$$

Now if we recall initial conditions of (10) in the form $X(0)=I_n$, $Z(h)=0$ and (8), we get $\sum_{k=1}^{+\infty}\left(X(kh)-X(kh-0)\right)=0$. Then using the first identity of Lemma 1, we can rewrite (24) as

$$\left(\breve{X}(p)B_1+\breve{Y}(p)B_2\right)\breve{u}(p)=\sum_{k=0}^{+\infty}\frac{1}{p^{k+1}}\sum_{j=0}^{+\infty}X_{k+1}(jh)e^{-jph}\breve{u}(p).$$

It follows from here, by returning to the original, that (10) can be represented as (15).

Similarly, considering the initial conditions of (11) in the form $X(0)=0$, $Z(h)=I_m$, we obtain

$$\left(X(0)+\sum_{k=1}^{+\infty}\left(X(kh)-X(kh-0)\right)e^{-pkh}\right)=$$
$$=\sum_{k=1}^{+\infty}\left(A_{22}e^{-ph}\right)^{k-1}A_{21}e^{-ph}=\left(I_m-A_{22}e^{-ph}\right)^{-1}A_{21}e^{-ph}\,.$$

If we combine this with (24) and the second identity of Lemma 1, we get

$$\left(\breve{X}(p)B_1+\breve{Y}(p)B_2\right)\breve{u}(p)=\sum_{k=0}^{+\infty}\frac{1}{p^{k+1}}\sum_{j=0}^{+\infty}Y_{k+1}(jh)e^{-jph}\breve{u}(p).$$

Finally, returning to the original, we come to (16). This completes the proof of Theorem 4.

## 6 Controllability, observability, duality

*Definition 1.* System (2), (3) is said to be:

i) $\mathbb{R}^n$ – controllable in $x$ on $[t_0, t_*]$ if for any $x_0, x_* \in \mathbb{R}^n$ and for any piecewise continuous m vector function $\psi(\tau), \tau\in[t_0, t_0+h]$, there exists a piecewise continuous control $u(\tau), \tau \in [t_0, t_*]$, such that

$$x(t_*; t_0, x_0, \psi, u) = x_* \quad (25)$$

ii) $\mathbb{R}^n$ – reachable in $x$ on $[t_0, t_*]$ if (25) holds for $x_0 = 0$ and $\psi(\tau) = 0, \tau \in [t_0, t_0 + h]$.

The notion of $\mathbb{R}^n$ – controllability in $y$ can be defined similarly.

Consider the following adjoint system $(t_* > t_0 + h)$

$$\frac{d}{d\tau} x^*(\tau) + x^*(\tau) A_{11}(\tau) + y^*(\tau) A_{21}(\tau) = 0,$$
$$y^*(\tau - h) - x^*(\tau) A_{12}(\tau) - y^*(\tau) A_{22}(\tau) = 0, \ \tau \in [t_0, t_*]. \quad (26)$$

with initial conditions

$x^*(t_* - 0) = x^*(t_*) = x_*, y^*(\tau) = \psi_*(\tau), \tau \in (t_* - h, t_*]$ and the output

$$z^*(t) = z^*(t; t_*, x_*, \psi_*) =$$
$$= x^*(\tau) B_1(\tau) + y^*(\tau) B_2(\tau), \ \tau \in [t_0, t_*] \ , \quad (27)$$

where $x^*(\tau) \in \mathbb{R}^{1\times n}, y^*(\tau) \in \mathbb{R}^{1\times m}, x_* \in \mathbb{R}^{1\times n}, \psi_*$ is a piecewise continuous m-row function.

*Definition 2.* System (26), (27) is said to be $\mathbb{R}^n$ – observable in x on $[t_0, t_*]$ if

$$z^*(t; t_* x_*, \psi_*) = z^*(t; t_*, \widetilde{x_*}, \psi_*) \ a.e.\ t \in [t_0, t_*] \Rightarrow x_* = \widetilde{x_*} \ .$$

Lemma. 4. Along with trajectories of the basic System (2) and adjoint Systems (26), (27), the following duality correlation is valid:

$$x^*(t_* - 0) x(t_* - 0) = x^*(t_0 + 0) x(t_0 + 0) +$$
$$+ \int_{t_0}^{t_0+h} y^*(t-h)\psi(t)dt - \int_{t_*}^{t_*+h} y^*(t-h)dt + \int_{t_0}^{t_*} z^*(t) u(t) dt \ .$$

*Proof.* We have

$$x^*(t_* - 0) x(t_* - 0) - x^*(t_0 + 0) x(t_0 + 0) =$$
$$= \int_{t_0}^{t_*} \frac{d}{dt}(x^*(t)x(t))dt = \int_{t_0}^{t_*} \dot{x}^*(t)x(t)dt + \int_{t_0}^{t_*} x^*(t)\dot{x}(t)dt =$$
$$= \int_{t_0}^{t_*} \dot{x}^*(t)x(t)dt + \int_{t_0}^{t_*} x^*(t)(A_{11}(t)x(t) + A_{12}(t)y(t) +$$
$$+ B_1(t)u(t))dt + \int_{t_0}^{t_*} y^*(t)\left(-y(t+h) + A_{21}(t)x(t) + \right.$$
$$\left. + A_{22}(t)y(t) + B_2(t)u(t)\right)dt =$$
$$= \int_{t_0}^{t_*} (\dot{x}^*(t) + x^*(t)A_{11}(t) + y^*(t)A_{21}(t))x(t)dt +$$
$$+ \int_{t_0}^{t_*} (-y^*(t-h) + x^*(t)A_{12}(t) + y^*(t)A_{22}(t))y(t)dt +$$
$$+ \int_{t_0}^{t_0+h} y^*(t-h)\psi(t)dt - \int_{t_*}^{t_*+h} y^*(t-h)y(t) + \int_{t_0}^{t_*}\left(x^*(t)B_1(t) + \right.$$
$$\left. + y^*(t)B_2(t)\right)u(t)dt = \int_{t_0}^{t_0+h} y^*(t-h)\psi(t)dt -$$
$$- \int_{t_*}^{t_*+h} y^*(t-h)y(t)dt + \int_{t_0}^{t_*} z^*(t)u(t)dt.$$

That finishes the proof.

*Theorem 5.* The following statements are equivalent:

i) System (2) is $\mathbb{R}^n$ – controllable in $x$ on $[t_0, t_*]$;

ii) System (2) is $\mathbb{R}^n$ – reachable in $x$ on $[t_0, t_*]$;

iii) The matrix rows

$X^*(t_*, \tau)B_1(\tau) + Y^*(t_*, \tau)B_2(\tau), \tau \in [t_0, t_*]$ are linearly independent a.e. in $[t_0, t_*]$;

iv) System (26), (27) is $\mathbb{R}^n$ – observable in $x$ on $[t_0, t_*]$.

*Sketch of the proof.* The equivalence i) $\Leftrightarrow$ ii) follows from linearity of System (2), in particular, taking into account the formula for solution representations. To prove ii) $\Leftrightarrow$ iii) observe, by the solution representation, in Theorem 1 that $\mathbb{R}^n$ – attainability of System (2) in $x$ on $[t_0, t_*]$ is equivalent to the implication:

$g \in \mathbb{R}^n, g^T(X^*(t_*, \tau)B_1(\tau) + Y^*(t_*, \tau)) = 0 \ a.e.\ t \in [t_0, t_*]$

$\Rightarrow g = 0 \in \mathbb{R}^n$ and, as a result, ii) $\Leftrightarrow$ iii) is established. Let us now prove ii) $\Leftrightarrow$ iv).First of all, System (2) is $\mathbb{R}^n$ – attainable with respect to $x$ on $[t_0, t_*]$ if and only if the implication holds:

$$g \in \mathbb{R}^n, g^T x(t_*, 0, 0, u) = 0 \ \ for\ \forall u \Rightarrow g = 0 \in \mathbb{R}^n$$

or, putting $x_* = g^T, \psi_*(\tau) = 0, \tau \in [t_* - h, t_*]$, and taking into account the duality correlation, we have

$$\int_{t_0}^{t_*} z^*(t; t_*, x_*, 0)u(t)dt = 0, \forall u \Rightarrow x_* = 0 \in \mathbb{R}^n$$

that is equivalent to $Z^*(t; t_*, x_*, 0) = 0 \ a.e.\ t \in [t_0, t_*]$. This, by linearity of the adjoint system, is equivalent to $\mathbb{R}^n$ – observability in $x$ on $[t_0, t_*]$ of System (26), (27). The proof is complete.

## 7 Parametric criteria for controllability and observability of stationary hybrid systems

In this section, along with system (2)–(4), we consider the following adjoint system

$$\dot{x}^*(t) = x^*(t)A_{11} + y^*(t)A_{21},$$
$$y^*(t+h) = x^*A_{12} + y^*(t)A_{22}, \qquad (28)$$
$$x(+0) = x(0) = x_0^*, y(\tau) = \psi^*(\tau), \tau \in [-h,0),$$

with the output

$$z(t) = z(t; x_0^*, \psi^*) = x^*(t)B_1 + y^*(t)B_2 . \qquad (29)$$

*Definition 3.* System (2)–(4) is called $t_1$-controllable for $t_1 > h$ if for any vector $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in \mathbb{R}^{n+m}$ and for any initial conditions (3) there exists a piecewise continuous control $u(\cdot)$ such that the condition $\begin{bmatrix} x(t_1) \\ y(t_1) \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ holds for the corresponding solution $x(t),\ y(t)$ of the system.

*Theorem 4.* System (2) is relatively $t_1$-controllable if and only if

$$rank \left[ \begin{bmatrix} X_k(ih) \\ Y_k(ih) \end{bmatrix}, k=0,1,...,n; i=0,1,...,\min\{T_{t_1-0}, m\} \right] = (m+n),$$

where $T_{t_1-0} = \lim\limits_{\varepsilon \to +0} \left[ \dfrac{t_1 - \varepsilon}{h} \right]$.

*Theorem 5.* System (2) is relatively $t_1$-controllable in $x$ if and only if

$$rank \left[ X_k(ih), k=1,2,...,n;\ i=0,1,...,\min\{T_{t_1-0}, m\} \right] = n.$$

*Theorem 6.* System (2) is relatively $t_1$-controllable in $y$ if and only if

$$rank \left[ Y_k(ih), k=0,1,...,n;\ i=0,1,...,\min\{T_{t_1-0}, m\} \right] = m.$$

*Definition 4.* System (2)–(4) is called $\mathbb{R}^n$ – controllable in $x$ if it is $\mathbb{R}^n$ – controllable with respect to $x$ on $[t_0, t_*]$ for some $t_* > t_0$ .

*Definition 5.* System (28), (29) is $\mathbb{R}^n$ – observable with respect to $x$ if

$$Z(t; x_0^*, \psi^*) = Z(t, \tilde{x}_0^*, \psi^*) \ \ a.e.\ t \in [t_0, t_*] \Rightarrow x_0^* = \tilde{x}_0^*.$$

*Theorem 6*. The following statements are equivalent:

i) System (2)-(4) is $\mathbb{R}^n$ – controllable in $x$;

ii) System (28), (29) is $\mathbb{R}^n$ – observable in $x$;

iii) $rank \left[ X_k(ih), k=1,2,...,n;\ i=0,1,...,\mathrm{m} \right] = n$ .

## 8 Concluding remarks

In the paper we have obtained definite integral and series representations of solutions for the hybrid difference-differential systems in symmetric form. The results considered have been applied to obtain parametric criteria for several types of relative controllability and observability in the case of stationary systems. As a result, the duality principle is formulated. Methods used can be generalized to more general controllability problems for the following ones. For given $p \geq 0$, $s > p$, and $s > 0$ System (2) is said to be $\mathbb{R}^n - (s,\ p)$-controllable in $x$ at time $t_* = t_0 + s$ if for any initial data $\varphi_0,\ \varphi$ and $\psi_0,\ \psi$ and for any piecewise continuous $r$-vector function $v$ there exists a piecewise continuous control function $u = u(\cdot)$ such that for the corresponding solutions the following condition holds

$$x(t_*;\ t_* - s,\ \varphi_0,\ \varphi,\ u) = x(t_*; t_* - p,\ \psi_0,\ \psi,\ v) .$$


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